Induction and Primitive Recursion in a Resource Conscious Logic — With a New Suggestion of How to Assign a Measure of Complexity to Primitive Recursive Functions

Uwe Petersen

Abstract. In [22], I presented a general approach to the definition of primitive recursive functions on the basis of a higher order logic without contraction employing a new kind of infinitary inference, the **Z**-inferences. The present paper is essentially a rewriting of this approach based on fixed-point constructions for the primitive recursive functions and a particular concern for the number of **Z**-inferences involved in proving results such as the recursion equations of primitive recursive functions and their totality.

1. Introduction

> Ever since the recursive functions have been identified there was a challenge to measure their inherent computational complexity, or in Kleene's words ⟦[15]⟧ to "classify the recursive functions into a hierarchy, according to some general principle".[1]

The present paper can be seen as a somewhat outlandish attempt to contribute to the problem of classifying primitive recursive functions.[2] It is based on a treatment of induction within a type free logic where "type free logic" is here used in the sense of [1] to mean that a logic "does not only possess formally the property of freedom of types, but beyond that an unrestricted axiom of comprehension."[3] Since the meaning of "unrestricted axiom of comprehension" may leave room for interpretation

[1] [25], p. 534.

[2] "Outlandish" in the sense that the author's primary research interest is dialectic in the Hegelean tradition and the ideas underlying the present contribution come out of that framework.

[3] On p. 3; my translation.

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(regarding the implication involved) I wish to specify that I require rules of the form

$$\frac{\mathfrak{F}[s]}{s \in \lambda x\, \mathfrak{F}[x]} \qquad \text{and} \qquad \frac{s \in \lambda x\, \mathfrak{F}[x]}{\mathfrak{F}[s]}$$

to be at least admissible, if not derivable.[4]

The basic idea for the type free logic employed here is to sacrifice contraction in exchange for unrestricted abstraction. Logic without contraction has this endearing feature to its credit that it allows a cut elimination proof which does not make recourse to the complexity of the cut formula. This is what makes logic without contraction such an ideal candidate for an underlying logic of a type free theory: the unpredictable way in which abstraction may change the complexity of the cut formula is irrelevant to a proof of cut elimination. It is also what has recently made it attractive to theoretical computer scientists in their quest for a "logic of polytime".[5]

But higher order logic without contraction is also a promising basis for a logical foundation in the style envisaged by Frege, no longer marred by inconsistencies. In other words, it is possible to take up again the reductionist approach in the foundations of mathematics after it has been cleared of the danger of antinomies stemming from unrestricted abstraction.[6]

The traditional way (Dedekind) of defining primitive recursive functions in a higher order logic follows the schema of induction. In the case of addition it commonly looks somewhat like this:

$$(1.1) \quad \mathcal{A} :\equiv \lambda x_1 x_2 x_3 \bigwedge y\,(\langle\langle x_1, 0\rangle, x_1\rangle \in y \wedge \bigwedge z_1 \bigwedge z_2\,(\langle\langle x_1, z_1\rangle, z_2\rangle \to \langle\langle x_1, z_1'\rangle, z_2'\rangle) \to \langle\langle x_1, x_2\rangle, x_3\rangle \in y)$$

with $s + t :\equiv \mathcal{A}[\![s, t]\!]$. I have taken this road in [21], sections 137a & 137b and repeated in [22], section 5.

[4] Type free logics of the kind presented in [2] and [6] are not "type-free" in this (strong) sense.

[5] Cf. [8].

[6] To be sure, this is not the only reduction that looks promising. Having gone through the experience of running into antinomies, higher logic now shares with metaphysics what Kant called the *Dialectic of Pure Reason* and my hope would be that metaphysics in turn can profit a bit from the methods that have been developed in foundational studies of mathematics and logic.

A certain "impredicativity" comes in here through the bound variable y being ruled by generalization. Employing fixed points this can be achieved "cheaper". Addition is declared as the fixed point $\mathcal{A}$ satisfying:

$$(1.2) \quad \mathcal{A} = \lambda x_1 x_2 x_3 ((x_2 = 0 \wedge x_3 = x_1) \vee \bigvee y_1 \bigvee y_2 \bigvee y_3 (y_1 = x_1 \sqcap y_2' = x_2 \sqcap y_3' = x_3 \sqcap \langle\langle y_1, y_2 \rangle, y_3 \rangle \in \mathcal{A}) .$$

Here the recursive character (calling itself up) comes from an unabashed application of the fixed point property.

Very little is actually needed on top of unrestricted abstraction to be able to prove a general fixed point property for term-forms $\mathfrak{F}$

$$(1.3) \qquad f = \lambda x \, \mathfrak{F}[f, x]$$

and what is needed is not lost by giving up contraction.[7]

With 1.3 at hand, terms for primitive recursive functions can be introduced according to 1.2 rather than the more traditional "second order" style indicated in 1.1 and thereby save a little bit on inductions. But when it comes to proving recursion equations and totality, some form of induction is indispensable. If one is only interested in a numerical representation, meta-theoretical inductions are sufficient. But for a proof of recursion equations with proper variables, induction on the formal level is required.

Due to the lack of contractions, however, special methods have to be introduced to achieve what can usually by accomplished by induction. Since the consistency of higher order logic without contraction is provable by ordinary induction, it will be clear that induction cannot be provable on the formal level. Actually, induction in its classical form can be shown to be incompatible with $\mathbf{L^iD}_\lambda$.[8]

In a higher order logic, induction is provided by a term of the form

$$\lambda x \bigwedge y (\bigwedge z (z \in y \to z' \in y) \to (0 \in y \to x \in y)) .$$

[7] The first clear statement of this (for the case of a type free logic) seems to be in [8], p. 173, proposition 4. Note, however, that Girard reserves the symbol = for identity (for which I use ≡) which is why his formulation looks slightly different. Cf. also [26], theorem 2, [20], p. 382, [5], p. 357, [19], theorem 10. In [20], lemma 7.2, I employed a notion of *application* (cf. definitions 2.4 below) which resulted in a more roundabout way of proving the fixed point property. Employing the notion of *co-domain of a relation* instead simplifies matters (cf. also [24], p. 122).

[8] Cf. [24], section 10.

This, however, no longer works without contraction: all one gets is that 0 and $0'$ fall under that term. What is lacking is the possibility of repeating the "induction step"

$$\bigwedge z\,(z \in t \to z' \in t)$$

ad libitum without having to "pay extra for it". In classical logic a wff includes a "use-as-often-as-you-like" license, and that by virtue of the axioms for implication.[9]

Desirable would be a way of expressing that assumptions can be used more often that once, but that this has to be accounted for.[10] In classical logic assumptions can always be used more often than just once, but one is not required to keep track of multiple uses.

Induction on the basis of classical logic is cheating: the problem of articulating "how many" doesn't arise thanks to contraction. Frege's analysis was more focused on the issue of a number being an equivalence class, than on the problem of how one can establish that 3, for instance, is a natural number without using the step (adding 1) three times. In a logic without contraction the notion of number is strongly tied to being able to repeat a particular operation, *viz.*, the application of the successor operation.

It is in these special methods that a kind of complexity comes in which is completely absent from a classical approach: keeping track of assumptions (resource consciousness).

Now I cannot claim to feel at home in the area of computational complexity nor do I feel confident to enter the discussion. However, engaging with the problem of recovering induction and recursion in a contraction free logic with unrestricted abstraction, I found myself placed in the neighborhood of questions concerning the possibility of classifying the recursive functions into a hierarchy. But, as I indicated in the introduction to this paper, my suggestion is a strange ("maverick") one, at least from a classical perspective: it is intimately linked to the way I introduced induction in [20] and employed it in [22]. This, in turn, cannot be separated from my way of treating infinity, *viz.*, through the introduction of **Z**-inferences. It

[9] That there is a problem with implication in a type free logic has long been observed. Paradoxes of the kind usually attributed to Curry require a restriction of implication which makes it impossible to obtain from the above formulation what is required for implication.

[10] This is what makes the logic "resource conscious": recycling of assumptions comes at a cost.

is the number of **Z**-inferences necessary to prove a result that will provide a measure for complexity.[11]

One last word before I close this introduction. The work presented here is of extremely basic nature and presenting deductions may well be regarded as a trivial exercise. But the point is to track down inferences that account for certain "totalizations", as I am inclined to call them, which I hope can provide a measure of complexity. In the course of trying to do this, I have made so many mistakes, mostly by being caught in a classical way of thinking, that I decided it would be better, at least for me, to write down deductions in virtually full length. This will enable those who are prepared to take the trouble of ploughing through details to see where I might have gone wrong and whether it will invalidate my project.

2. Basic notions

In this section I mainly repeat definitions and provide a few basic results that have been established in [20] and, above all, in [21].[12]

DEFINITIONS 2.1. (1) Primitive symbols:

(1.1) symbols for free and bound *variables*: a, b, c, and x, y, z, also with index numbers;

(1.2) the *constants* λ, $\in$, $\bigwedge$, $\rightarrow$, and $\Box$.

(2) The language $\mathcal{L}$ is defined accordingly.

REMARK 2.2. This is not the most economical choice of primitive symbols, but rather an attempt at making more accessible considerations regarding the notion of Z-specific wffs introduced in [20], p. 388.

DEFINITIONS 2.3. (1) Initial sequents: $A \Rightarrow A$.
(2) Structural rules:

$$\text{Weakening :} \quad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,, \qquad \text{Exchange :} \quad \frac{A, \Gamma \Rightarrow \mathfrak{F}[s]}{\Gamma \Rightarrow s \in \lambda x\, \mathfrak{F}[x]}\,,$$

[11] It should be clear that **Z**-inferences are not needed in order to prove numerical results like $3 + 2 = 5$, for instance.

[12] Some of the following definitions may slightly differ from the ones I have given elsewhere. It should be clear, however, that they are logically equivalent to the ones given there, if not explicitly stated otherwise.

$$\text{Cut}: \quad \frac{\Gamma \Rightarrow A \qquad A, \Pi \Rightarrow}{\Gamma, \Pi \Rightarrow B}\ .$$

(3) Operational rules.
(3.1) Rules for $\in$:

$$\text{left}: \quad \frac{\mathfrak{F}[s], \Gamma \Rightarrow C}{s \in \lambda x\, \mathfrak{F}[x], \Gamma \Rightarrow C}\ ; \qquad \text{right}: \quad \frac{A, \Gamma \Rightarrow \mathfrak{F}[s]}{\Gamma \Rightarrow s \in \lambda x\, \mathfrak{F}[x]}\ .$$

(3.2) Rules for $\bigwedge$:

$$\text{left}: \quad \frac{\mathfrak{F}[s], \Gamma \Rightarrow C}{\bigwedge y\, \mathfrak{F}[y], \Gamma \Rightarrow C}\ ; \qquad \text{right}: \quad \frac{\Gamma \Rightarrow \mathfrak{F}[a]}{\Gamma \Rightarrow \bigwedge y\, \mathfrak{F}[y]}\ .$$

with the usual condition on the *eigenvariable* a.
(3.3) Rules for $\rightarrow$:

$$\text{left}: \quad \frac{\Gamma \Rightarrow A \qquad B, \Pi \Rightarrow C}{A \rightarrow B, \Gamma, \Pi \Rightarrow C}\ ; \qquad \text{right}: \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\ .$$

(3.4) Rules for $\square$:

$$\text{left}: \quad \frac{A, B, \Gamma \Rightarrow C}{A \square B, \Gamma \Rightarrow C}\ ; \qquad \text{right}: \quad \frac{\Gamma \Rightarrow A \qquad \Pi \Rightarrow B}{\Gamma, \Pi \Rightarrow A \square B}\ .$$

(4) The formalized theory $\mathbf{L^iD}_\lambda$ is defined as the language $\mathcal{L}$ with the foregoing initial sequents and rules of inference.

DEFINITIONS 2.4. The following is a list of defined constants:[13]

$$\begin{aligned}
s \subseteq t &:\equiv \bigwedge x (x \in s \rightarrow x \in t)\,; \\
\mathcal{V} &:\equiv \lambda x\, (x \subseteq x)\,; \\
\bot &:\equiv \bigwedge x\, (\mathcal{V} \subseteq x)\,; \\
\neg A &:\equiv A \rightarrow \bot\,; \\
\emptyset &:\equiv \lambda \bot\,; \\
\top &:\equiv \neg \bot\,; \\
\bigvee x\, \mathfrak{F}[x] &:\equiv \bigwedge y (\bigwedge x (\mathfrak{F}[x] \rightarrow \lambda \top \subseteq y) \rightarrow \lambda \top \subseteq y) \quad \text{(existence)}\,; \\
s \equiv t &:\equiv \bigwedge y\, (s \in y \rightarrow t \in y) \qquad \text{(identity)}\,;
\end{aligned}$$

[13] This list is in large parts identical to that in [20], p. 66 f. It is provided here for the sake of convenience.

$$
\begin{aligned}
A \mathbin{\square} B &:\equiv \bigwedge x\,((A \to (B \to \lambda\top \subseteq x)) \to \lambda\top \subseteq x)\,; \\
A \wedge B &:\equiv \bigwedge x\,(\bigwedge y\,(\lambda A \in y \to (\lambda B \in y \to x \in y)) \to \lambda\top \subseteq x)\,; \\
A \vee B &:\equiv \bigwedge y\,((A \to \lambda\top \subseteq y) \wedge (B \to \lambda\top \subseteq y) \to \lambda\top \subseteq x)\,; \\
\lBrace s \rBrace &:\equiv \lambda x\,(s \equiv x) \qquad (\text{“exclusive” singleton})\,; \\
\lBrace s, t \rBrace &:\equiv \lambda x\,(x \equiv s \vee x \equiv t) \qquad (\text{“exclusive” pairing})\,; \\
A \leftrightarrow B &:\equiv (A \to B) \wedge (B \to A)\,; \\
s = t &:\equiv \bigwedge x\,(x \in s \leftrightarrow x \in t) \qquad (\text{equality})\,; \\
\{s\} &:\equiv \lambda x\,(x = s) \qquad (\text{“inclusive” singleton})\,; \\
\{s, t\} &:\equiv \lambda x\,(x = s \vee x = t) \qquad (\text{“inclusive” pairing})\,; \\
\langle s, t \rangle &:\equiv \{\{\{s\}, \emptyset\}, \{\{t\}\}\} \qquad (\text{“inclusive” ordered pair})\,; \\
\llangle s, t \rrangle &:\equiv \lBrace \lBrace \lBrace s \rBrace, \emptyset \rBrace, \lBrace \lBrace t \rBrace \rBrace \rBrace \qquad (\text{“exclusive” ordered pair})\,; \\
\lambda xy\, \mathfrak{F}[x, y] &:\equiv \lambda z \bigvee x \bigvee y\,(z \equiv \llangle x, y \rrangle \mathbin{\square} \mathfrak{F}[x, y]) \qquad (\text{dyadic abstract})\,; \\
s \cup t &:\equiv \lambda x\,(x \in s \vee x \in t)\,; \\
s \sqcap t &:\equiv \lambda x\,(x \in s \mathbin{\square} x \in t)\,; \\
st &:\equiv \lambda x(\llangle t, x \rrangle \in s) \qquad (\text{co-domain of a relation})\,; \\
s[\![t]\!] &:\equiv \lambda x \bigwedge y(\llangle t, y \rrangle \in s \to x \in y) \qquad (\text{application})\,.
\end{aligned}
$$

I repeat a few notational conventions from [21].

CONVENTIONS 2.5. (1) $[A]^{\cdot 2} :\equiv A \mathbin{\square} A$.
(2) $[A/s] :\equiv \lambda A \in s$.
(3) $\mathrm{k}\,[A]$ is inductively defined as follows:
 (3.1) $1\,[A] :\equiv A$;
 (3.2) $\mathrm{k}'\,[A] :\equiv A, \mathrm{k}[A]$.

Before being able to express induction over numbers, I need a way of expressing that an assumption may be used a certain number of times. The next definition provides some basic ingredients.

DEFINITIONS 2.6. (1) $I :\equiv \lambda x(x = \mathcal{V})$, *i.e.*, $\{\mathcal{V}\}$.
(2) $s^I :\equiv \lambda x(x \in s \mathbin{\square} x \in I)$, *i.e.*, $s \sqcap I$.

Definitions 2.6 allow us to express (and prove) simple properties such as $[A/I^I] \leftrightarrow A \mathbin{\square} A$.

Proposition 2.7. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(2.7i) $[A/I] \leftrightarrow A$;

(2.7ii) $[A/s^I] \leftrightarrow [A/s] \mathbin{\square} A$;

(2.7iii) $[A/s \sqcap t] \Rightarrow [A/s] \mathbin{\square} [A/t]$;

(2.7iv) $[A/s], [A/t] \Rightarrow [A/s \sqcap t]$;

(2.7v) $s = t, [A/s] \Rightarrow [A/t]$.

Proof. Cf. [21], p. 1792. QED

Definition 2.8. $s \sqcap^{\mathrm{k}} t$ is defined inductively as follows:

(1) $s \sqcap^1 t :\equiv s \sqcap t$;
(2) $s \sqcap^{\mathrm{k}'} t :\equiv (s \sqcap^{\mathrm{k}} t) \sqcap t$.

Proposition 2.9. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(2.9i) $\Rightarrow (s \sqcap t)^{II} = (s^I \sqcap t^I)$;

(2.9ii) $\Rightarrow ((s \sqcap t) \sqcap r)^{III} = (s^I \sqcap t^I) \sqcap r^I$;

(2.9iii) $\Rightarrow (s \sqcap^{\mathrm{k}} r) \sqcap (t \sqcap^{\mathrm{k}} r) = (s \sqcap t) \sqcap^{2\mathrm{k}} r$.

Proof. These are straightforward consequences of the associativity and commutativity of $\sqcap$; left to the reader. QED

Definitions 2.10. (1) The (intuitive) set $\mathbf{\Psi}$ is defined inductively as follows:

(1.1) I is an element of $\mathbf{\Psi}$;
(1.2) If s is an element of $\mathbf{\Psi}$, then so is s^I.

(2) If $\mathrm{m} \in \mathbb{N}\backslash\{0\}$, then its *corresponding* $\mathbf{\Psi}$*-element* is defined inductively as follows:

(2.1) I is the *corresponding* $\mathbf{\Psi}$*-element* of 1.
(2.1) If $\tilde{n}$ is the corresponding $\mathbf{\Psi}$-element of n, the $\tilde{n}^I$ is the *corresponding* $\mathbf{\Psi}$*-element* of n'.

Remark 2.11. In the appendix, section 12 at the end of this paper, this correspondence between natural numbers > 0 and elements of $\mathbf{\Psi}$ will be established on the formal level.

Definitions 2.12. (1) $\breve{\gamma}[A] :\equiv \lambda x((x \in x) \sqcap A) \in \lambda x((x \in x) \sqcap A)$.[14]
(2) $\mathbf{Z} :\equiv \lambda x \bigwedge y(\breve{\gamma}[\bigwedge z(z \in y \to z^I \in y)] \to (I \in y \to x \in y))$.

I list a few properties of $\breve{\gamma}$.

Proposition 2.13. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(2.13i) $\breve{\gamma}[A] \Rightarrow \breve{\gamma}[A] \sqcap A$;

(2.13ii) $\breve{\gamma}[A] \Rightarrow A$;

(2.13iii) $\breve{\gamma}[A \to B], \breve{\gamma}[A] \Rightarrow B$;

(2.13iv) $\breve{\gamma}[A] \to (A \to B), \breve{\gamma}[A] \Rightarrow B$.

Proof. Cf. [21], p. 1804. QED

Proposition 2.14. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(2.14i) $\Rightarrow I \in \mathbf{Z}$;

(2.14ii) $s \in \mathbf{Z} \Rightarrow s^I \in \mathbf{Z}$;

(2.14iii) $s \in \mathbf{Z} \Rightarrow (s \sqcap^{\mathrm{k}} I) \in \mathbf{Z}$.

Proof. As regards 2.14i and 2.14ii, cf. [21], p. 1806. 2.14iii is obtained by a straightforward induction on k which is left to the reader. QED

Definitions 2.15.
(1) $\breve{\mathbf{\Pi}}^\circ :\equiv \lambda x(x \in \mathbf{Z} \sqcap \bigwedge y([I \in y \wedge \bigwedge z(z \in y \to z^I \in y)/x] \to x \in y))$.
(2) $\Box A :\equiv \bigwedge x(x \in \breve{\mathbf{\Pi}}^\circ \sqcap [A/x])$.
(3) $A \supset B :\equiv \bigvee x(x \in \breve{\mathbf{\Pi}}^\circ \sqcap ([A/x] \to B))$.
(4) $\breve{\mathbf{\Pi}} :\equiv \lambda x \bigvee y(y \in \breve{\mathbf{\Pi}}^\circ \sqcap y = x)$.[15]

Proposition 2.16. *Inferences according to the following schema are* $\mathbf{L^iD}_\lambda$*-derivable:*

$$\frac{B, A, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow A \supset C} .$$

[14] This is just an explicit fixed point construction.

[15] Now that *is* different from the one in [20], p. 398, but hopefully no reason for concern. Cf. also footnote 22 below.

Proof.

$$\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow a^{I}\in\breve{\mathbf{\Pi}}^{\circ} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A/a]\Rightarrow[A/a] \qquad B,A,\mathit{\Gamma}\Rightarrow C}{[A/a]\to B,[A/a],A,\mathit{\Gamma}\Rightarrow C}}{[A/a]\to B,[A/a^{I}],\mathit{\Gamma}\Rightarrow C}}{[A/a]\to B,\mathit{\Gamma}\Rightarrow[A/a^{I}]\to C}}{a\in\breve{\mathbf{\Pi}}^{\circ},[A/a]\to B,\mathit{\Gamma}\Rightarrow a^{I}\in\breve{\mathbf{\Pi}}^{\circ}\square([A/a^{I}]\to C)}}{a\in\breve{\mathbf{\Pi}}^{\circ}\square([A/a]\to B),\mathit{\Gamma}\Rightarrow a^{I}\in\breve{\mathbf{\Pi}}^{\circ}\square([A/a^{I}]\to C)}}{a\in\breve{\mathbf{\Pi}}^{\circ}\square([A/a]\to B),\mathit{\Gamma}\Rightarrow\bigvee x(x\in\breve{\mathbf{\Pi}}^{\circ}\square([A/x]\to C))}}{\dfrac{\bigvee x(x\in\breve{\mathbf{\Pi}}^{\circ}\square([A/x]\to B)),\mathit{\Gamma}\Rightarrow\bigvee x(x\in\breve{\mathbf{\Pi}}^{\circ}\square([A/x]\to C))}{A\supset B,\mathit{\Gamma}\Rightarrow A\supset C}}\,.$$

QED

REMARK 2.17. The point of the foregoing result: the formulation of $\mathbf{N}^{\circ}$ in definition 4.4 below in terms of the weak implication $\supset$ does not make additional deductive power necessary when it comes to establishing $s\in\mathbf{N}^{\circ}\Rightarrow s'\in\mathbf{N}^{\circ}$ (4.6ii below).

PROPOSITION 2.18. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(2.18i) $\quad\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}$;

(2.18ii) $\quad s\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow s^{I}\in\breve{\mathbf{\Pi}}^{\circ}$;

(2.18iii) $\quad s\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow(s\sqcap^{\mathrm{k}}I)\in\breve{\mathbf{\Pi}}^{\circ}$.

Proof. As regards 2.18i and 2.18ii cf. 134.3iii and 134.3iv in [20], p. 1825. It must be clear though that these are indeed $\mathbf{L^{i}D}_{\lambda}$-deducible, *i.e.*, no **Z**-inference required.[16] 2.18iii as for 2.14iii. QED

3. Z-inferences

As it stands, $\breve{\mathbf{\Pi}}^{\circ}$ doesn't offer much of an advantage as against **Z**. This is now going to change with the introduction of **Z**-inferences.

[16] In [20], they were listed as $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$-deducible.

DEFINITIONS 3.1. (1) An inference according to the schema

$$\frac{\Gamma \Rightarrow s \in \mathbf{Z} \qquad \Rightarrow A}{\Gamma \Rightarrow [A/s]}$$

is called a **Z**-inference.[17]
(2) The formalized theory $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$ is obtained from $\mathbf{L^{i}D}_{\lambda}$ by adding all **Z**-inferences.

In what follows I shall mostly consider "throttled" versions of $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$.

DEFINITION 3.2. $\mathbf{L^{i}D}^{\mathrm{Z}\restriction \mathrm{n}}_{\lambda}$ is defined as $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$ with the restriction that a $\mathbf{L^{i}D}^{\mathrm{Z}\restriction \mathrm{n}}_{\lambda}$-deduction can contain at most n **Z**-inferences.

PROPOSITION 3.3. *Inferences according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$*-derivable with an increase of the* **Z***-grade indicated on the right:*

(3.3i)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[I] \qquad \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{I}]}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow \mathfrak{F}[s]}\ {\scriptstyle +1}\,;$$

(3.3ii)
$$\frac{\Gamma, \mathfrak{A}[I] \Rightarrow \mathfrak{B}[I] \qquad \mathfrak{A}[a] \to \mathfrak{B}[a], \mathfrak{A}[a^{I}] \Rightarrow \mathfrak{B}[a^{I}]}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma, \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}\ {\scriptstyle +1}\,;$$

(3.3iii)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[I] \qquad \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{I}] \qquad \mathfrak{F}[s], \Gamma, \Pi \Rightarrow C}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma, \Pi \Rightarrow C}\ {\scriptstyle +1}\,;$$

(3.3iv)
$$\frac{s \in \breve{\mathbf{\Pi}}^{\circ}, s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow C}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow C}\ {\scriptstyle +1}\,;$$

(3.3v)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[I] \qquad a \in \breve{\mathbf{\Pi}}^{\circ}, \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{I}]}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow \mathfrak{F}[s]}\ {\scriptstyle +2}\,.$$

[17] **Z**-inferences have been first introduced in [20], p. 392. I shall not here comment on the meta-theoretical side of these inferences but only refer curious readers to [21], section 119b, for a little bit of justification.

Proof. *Re* 3.3i.

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\dfrac{\Rightarrow \mathfrak{F}[I]}{\Rightarrow I \in \lambda x\, \mathfrak{F}[x]}
\quad
\dfrac{\dfrac{\dfrac{\mathfrak{F}[c] \Rightarrow \mathfrak{F}[c^I]}{\overline{c \in \lambda x\, \mathfrak{F}[x] \Rightarrow c^I \in \lambda x\, \mathfrak{F}[x]}}}{\Rightarrow c \in \lambda x\, \mathfrak{F}[x] \rightarrow c^I \in \lambda x\, \mathfrak{F}[x]}}{\Rightarrow \bigwedge z(z \in \lambda x\, \mathfrak{F}[x] \rightarrow z^I \in \lambda x\, \mathfrak{F}[x])}
}{
\dfrac{\Rightarrow I \in \lambda x\, \mathfrak{F}[x] \wedge \bigwedge z(z \in \lambda x\, \mathfrak{F}[x] \rightarrow z^I \in \lambda x\, \mathfrak{F}[x])}{a \in \mathbf{Z} \Rightarrow [I \in \lambda x\, \mathfrak{F}[x] \wedge \bigwedge z(z \in \lambda x\, \mathfrak{F}[x] \rightarrow z^I \in \lambda x\, \mathfrak{F}[x])/a]}\,{}^{+1}
}
\quad
\dfrac{\mathfrak{F}[s] \Rightarrow \mathfrak{F}[s]}{a \in \lambda x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[s]}
}{
\dfrac{\dfrac{\dfrac{a \in \mathbf{Z}, [I \in \lambda x\, \mathfrak{F}[x] \wedge \bigwedge z(z \in \lambda x\, \mathfrak{F}[x] \rightarrow z^I \in \lambda x\, \mathfrak{F}[x])/a] \rightarrow a \in \lambda x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[s]}{a \in \mathbf{Z}, \bigwedge y([I \in y \wedge \bigwedge z(z \in y \rightarrow z^I \in y)/a] \rightarrow a \in y) \Rightarrow \mathfrak{F}[s]}}{a \in \mathbf{Z} \,\Box \bigwedge y([I \in y \wedge \bigwedge z(z \in y \rightarrow z^I \in y)/a] \rightarrow a \in y) \Rightarrow \mathfrak{F}[s]}}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow \mathfrak{F}[s]}
}\;.
$$

Re 3.3ii. Essentially as for 3.3i. The point is to see that no cut (or inversion) is required. Let $\xi :\equiv \lambda x(\mathfrak{A}[x] \rightarrow \mathfrak{B}[x])$

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\mathfrak{A}[I] \Rightarrow \mathfrak{B}[I]}{\Rightarrow \mathfrak{A}[I] \rightarrow \mathfrak{B}[I]}}{\Rightarrow I \in \xi}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A}[a] \rightarrow [B/a], \mathfrak{A}[a^I] \Rightarrow [B/a^I]}{\mathfrak{A}[a] \rightarrow [B/a] \Rightarrow \mathfrak{A}[a^I] \rightarrow [B/a^I]}}{\overline{c \in \xi \Rightarrow c^I \in \xi}}}{\Rightarrow c \in \xi \rightarrow c^I \in \xi}}{\Rightarrow \bigwedge z(z \in \xi \rightarrow z^I \in \xi)}
}{
\dfrac{\Rightarrow I \in \xi \wedge \bigwedge z(z \in \xi \rightarrow z^I \in \xi)}{s \in \mathbf{Z} \Rightarrow [I \in \xi \wedge \bigwedge z(z \in \xi \rightarrow z^I \in \xi)/s]}\,{}^{+1}
}
\quad
\dfrac{\dfrac{\mathfrak{A}[s] \Rightarrow \mathfrak{A}[s] \qquad \mathfrak{B}[s] \Rightarrow \mathfrak{B}[s]}{\mathfrak{A}[s] \rightarrow \mathfrak{B}[s], \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}}{s \in \xi, \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}
}{
\dfrac{\dfrac{\dfrac{s \in \mathbf{Z}, [I \in \xi \wedge \bigwedge z(z \in \xi \rightarrow z^I \in \xi)/s] \rightarrow s \in \xi), \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}{s \in \mathbf{Z}, \bigwedge y([I \in y \wedge \bigwedge z(z \in y \rightarrow z^I \in y)/s] \rightarrow s \in y), \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}}{s \in \mathbf{Z} \,\Box \bigwedge y([I \in y \wedge \bigwedge z(z \in y \rightarrow z^I \in y)/s] \rightarrow s \in y), \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}}{s \in \breve{\mathbf{\Pi}}^{\circ}, \mathfrak{A}[s] \Rightarrow \mathfrak{B}[s]}
}\;.
$$

Re 3.3iii. Essentially as for 3.3ii; left to the reader. Cf. also 4.7ii below.

Re 3.3iv. This is a straightforward consequence of 3.3iii. Employ 2.18i and 2.18ii:

$$\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ} \qquad \Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}}{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}\Box I\in\breve{\mathbf{\Pi}}^{\circ}} \quad \dfrac{\dfrac{c\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow c^{I}\in\breve{\mathbf{\Pi}}^{\circ} \quad c\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow c^{I}\in\breve{\mathbf{\Pi}}^{\circ}}{c\in\breve{\mathbf{\Pi}}^{\circ}, c\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow c^{I}\in\breve{\mathbf{\Pi}}^{\circ}\Box c^{I}\in\breve{\mathbf{\Pi}}^{\circ}}}{c\in\breve{\mathbf{\Pi}}^{\circ}\Box c\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow c^{I}\in\breve{\mathbf{\Pi}}^{\circ}\Box c^{I}\in\breve{\mathbf{\Pi}}^{\circ}} \quad \dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}, s\in\breve{\mathbf{\Pi}}^{\circ}, \varGamma\Rightarrow C}{s\in\breve{\mathbf{\Pi}}^{\circ}\Box s\in\breve{\mathbf{\Pi}}^{\circ}, \varGamma\Rightarrow C}}{s\in\breve{\mathbf{\Pi}}^{\circ}, \varGamma\Rightarrow C}\,{+1}.$$

Re 3.3v. Employ 3.3iv and 3.3i:

$$\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ} \qquad \varGamma\Rightarrow\mathfrak{F}[I]}{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}\Box\mathfrak{F}[I]} \qquad \dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow a^{I}\in\breve{\mathbf{\Pi}}^{\circ} \qquad a\in\breve{\mathbf{\Pi}}^{\circ},\mathfrak{F}[a]\Rightarrow\mathfrak{F}[a^{I}]}{a\in\breve{\mathbf{\Pi}}^{\circ}, a\in\breve{\mathbf{\Pi}}^{\circ},\mathfrak{F}[a]\Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}\Box\mathfrak{F}[a^{I}]}}{a\in\breve{\mathbf{\Pi}}^{\circ},\mathfrak{F}[a]\Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}\Box\mathfrak{F}[a^{I}]}\,{+1}}{a\in\breve{\mathbf{\Pi}}^{\circ}\Box\mathfrak{F}[a]\Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}\Box\mathfrak{F}[a^{I}]}}{s\in\breve{\mathbf{\Pi}}^{\circ},\varGamma\Rightarrow\mathfrak{F}[s]}\,{+1}\,.$$

QED

REMARK 3.4. The reason for taking the detour *via* $\mathbf{Z}$ to get to $\breve{\mathbf{\Pi}}^{\circ}$ should become sufficiently clear by looking at the proof of 3.3i above. In view of its obvious similarity to induction, I shall occasionally refer to it as proto-induction.

PROPOSITION 3.5. *The following holds:*

(3.5i) $\mathbf{L^{i}D}_{\lambda}\vdash\ \Rightarrow (I\sqcap^{\mathrm{k}} I)\in\breve{\mathbf{\Pi}}\,;$

(3.5ii) $\mathbf{L^{i}D}_{\lambda}^{\mathrm{Z}\restriction 1}\vdash\ s\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow (s\sqcap^{\mathrm{k}} s)\in\breve{\mathbf{\Pi}}\,.$

Proof. Re 3.5i. This is an immediate consequence of 2.18iii.
Re 3.5ii. This is a straightforward application of 3.3i employing 2.9i:

$$\dfrac{\Rightarrow (I\sqcap^{\mathrm{k}} I)\in\breve{\mathbf{\Pi}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{b\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow (b\sqcap^{2\mathrm{k}} I)\in\breve{\mathbf{\Pi}}^{\circ} \quad \dfrac{\Rightarrow (c\sqcap^{\mathrm{k}} c)\sqcap^{2\mathrm{k}} I = (c^{I}\sqcap^{\mathrm{k}} c^{I})}{b=(c\sqcap^{\mathrm{k}} c)\Rightarrow (b\sqcap^{2\mathrm{k}} I)=(c^{I}\sqcap^{\mathrm{k}} c^{I})}}{b\in\breve{\mathbf{\Pi}}^{\circ}, b=(c\sqcap^{\mathrm{k}} c)\Rightarrow (b\sqcap^{2\mathrm{k}} I)\in\breve{\mathbf{\Pi}}^{\circ}\Box(b\sqcap^{2\mathrm{k}} I)=(c^{I}\sqcap^{\mathrm{k}} c^{I})}}{b\in\breve{\mathbf{\Pi}}^{\circ}, b=(c\sqcap^{\mathrm{k}} c)\Rightarrow (c^{I}\sqcap^{\mathrm{k}} c^{I})\in\breve{\mathbf{\Pi}}}}{b\in\breve{\mathbf{\Pi}}^{\circ}\Box b=(c\sqcap^{\mathrm{k}} c)\Rightarrow (c^{I}\sqcap^{\mathrm{k}} c^{I})\in\breve{\mathbf{\Pi}}}}{(c\sqcap^{\mathrm{k}} c)\in\breve{\mathbf{\Pi}}\Rightarrow (c^{I}\sqcap^{\mathrm{k}} c^{I})\in\breve{\mathbf{\Pi}}}}{s\in\breve{\mathbf{\Pi}}^{\circ}\Rightarrow (s\sqcap^{\mathrm{k}} s)\in\breve{\mathbf{\Pi}}}\,.$$

QED

Proposition 3.6. *Inferences according to the following schemata are* $\mathbf{L^iD^Z_\lambda}$*-derivable with an increase of the* $\mathbf{Z}$*-grade indicated on the right:*

(3.6i) $$\frac{\Rightarrow A}{s\in\breve{\mathbf{\Pi}}^\circ \Rightarrow [A/s]}\,{\scriptstyle +1}\,;$$

(3.6ii) $$\frac{\Rightarrow A}{\Rightarrow \Box A}\,{\scriptstyle +1}\,;$$

(3.6iii) $$\frac{A \Rightarrow B}{s\in\breve{\mathbf{\Pi}}^\circ, [A/s] \Rightarrow [B/s]}\,{\scriptstyle +1}\,;$$

(3.6iv) $$\frac{(\mathrm{k}+1)[A] \Rightarrow B}{s\in\breve{\mathbf{\Pi}}^\circ, [A/s \sqcap^{\mathrm{k}} s] \Rightarrow [B/s]}\,{\scriptstyle +1}\,;$$

(3.6v) $$\frac{\Rightarrow A \qquad B, \Pi \Rightarrow C}{A \supset B, \Pi \Rightarrow C}\,{\scriptstyle +1}\,;$$

(3.6vi) $$\frac{\Gamma \Rightarrow A \qquad B, \Pi \Rightarrow C}{A \supset B, \Box\Gamma, \Pi \Rightarrow C}\,{\scriptstyle +2}\,;$$

(3.6vii) $$\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,{\scriptstyle +2}\,;$$

(3.6viii) $$\frac{A, B \Rightarrow C}{\Box A, \Box B \Rightarrow \Box C}\,{\scriptstyle +2}\,;$$

(3.6ix) $$\frac{2[A] \Rightarrow B}{\Box A \Rightarrow \Box B}\,{\scriptstyle +2}\,;$$

(3.6x) $$\frac{\mathrm{k}[A] \Rightarrow B}{\Box A \Rightarrow \Box B}\,{\scriptstyle +2}\,;$$

(3.6xi) $$\frac{\Box(s\in\breve{\mathbf{\Pi}}^\circ), \Gamma \Rightarrow C}{s\in\breve{\mathbf{\Pi}}^\circ, \Gamma \Rightarrow C}\,{\scriptstyle +4}\,.$$

Proof. Re 3.6i. This is a straightforward application of 3.3i employing 2.7ii:

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow [A/I]} \qquad \dfrac{[A/a] \Rightarrow [A/a] \qquad \Rightarrow A}{[A/a] \Rightarrow [A/a^{I}]}}{s \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow [A/s]}\,{}^{+1}\ .$$

Re 3.6ii. Cf. 134.21ii in [21], p. 1834.
Re 3.6iii. Cf. 134.10ii in [21], p. 1830.
Re 3.6iv. I only show the case of k = 1. Employ 2.7iv:

$$\dfrac{\dfrac{A, A \Rightarrow B}{[A/I \sqcap I] \Rightarrow [B/I]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A/c], [A/c] \Rightarrow [A/c \sqcap c] \qquad [B/c] \Rightarrow [B/c]}{[A/c \sqcap c] \to [B/c], [A/c], [A/c] \Rightarrow [B/c]} \qquad A, A \Rightarrow B}{[A/c \sqcap c] \to [B/c], [A/c], [A/c], A, A \Rightarrow [B/c] \,\square\, B}}{[A/c \sqcap c] \to [B/c], [A/c^{I}], [A/c^{I}] \Rightarrow [B/c^{I}]}}{[A/c \sqcap c] \to [B/c], [A/c^{I} \sqcap c^{I}] \Rightarrow [B/c^{I}]}}{s \in \breve{\mathbf{\Pi}}^{\circ}, [A/s \sqcap s] \Rightarrow [B/s]}\,{}^{+1}\ .$$

Re 3.6v. Straightforward consequence of the definition of $\supset$ and 3.6i. Cf. 135.20iv in [21], p. 1847.
Re 3.6vi. Essentially as for 3.6v only with an additional inference according to schema 3.3iv. Left to the reader.
Re 3.6vii. Employ 3.3iv:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ} \qquad \dfrac{A \Rightarrow B}{[A/a], a \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}\,{}^{+1}}{a \in \breve{\mathbf{\Pi}}^{\circ} \to [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}, a \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}}{\bigwedge x (x \in \breve{\mathbf{\Pi}}^{\circ} \to [A/x]), a \in \breve{\mathbf{\Pi}}^{\circ}, a \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}}{\bigwedge x (x \in \breve{\mathbf{\Pi}}^{\circ} \to [A/x]), a \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}\,{}^{+1}}{\bigwedge x (x \in \breve{\mathbf{\Pi}}^{\circ} \to [A/x]) \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ} \to [B/a]}}{\bigwedge x (x \in \breve{\mathbf{\Pi}}^{\circ} \to [A/x]) \Rightarrow \bigwedge x (x \in \breve{\mathbf{\Pi}}^{\circ} \to [B/x])}\ .$$

Re 3.6viii. Employ 3.3iv:

$$\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ} \qquad \dfrac{a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ} \qquad \dfrac{A, B \Rightarrow C}{[A/a], [B/a], a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [C/a]}{\scriptstyle +1}}{[A/a], a\in\breve{\mathbf{\Pi}}^{\circ} \to [B/a], a\in\breve{\mathbf{\Pi}}^{\circ}, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [C/a]}}{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ} \to [A/a], a\in\breve{\mathbf{\Pi}}^{\circ} \to [B/a], a\in\breve{\mathbf{\Pi}}^{\circ}, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [C/a]}{\dfrac{\bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [A/x]), \bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [B/x]), a\in\breve{\mathbf{\Pi}}^{\circ}, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [C/a])}{\dfrac{\bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [A/x]), \bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [B/x]), a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [C/a])}{\dfrac{\bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [A/x]), \bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [B/x]) \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ} \to [C/a]}{\bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [A/x]), \bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [B/x]) \Rightarrow \bigwedge x(x\in\breve{\mathbf{\Pi}}^{\circ} \to [C/x])}}{\scriptstyle +1}}}}{\scriptstyle +1} \; .$$

Re 3.6ix. Employ 3.6iv and 3.3iii:[18] Let $\mathfrak{Q} :\equiv *_1 \in\breve{\mathbf{\Pi}}^{\circ} \mathbin{\scriptstyle\square} (*_1 \sqcap *_1)\in\breve{\mathbf{\Pi}}$

$$\dfrac{\dfrac{\Rightarrow \mathfrak{Q}[I] \qquad \mathfrak{Q}[c] \Rightarrow \mathfrak{Q}[c^I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{b\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow b\in\breve{\mathbf{\Pi}}^{\circ} \qquad \dfrac{\dfrac{A, A \Rightarrow B}{[A/a \sqcap a], a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}{\scriptstyle +1}}{b = a \sqcap a, [A/b], a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}}{b\in\breve{\mathbf{\Pi}}^{\circ}, b = a \sqcap a, b\in\breve{\mathbf{\Pi}}^{\circ} \to [A/b], a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}{\scriptstyle +1}}{b\in\breve{\mathbf{\Pi}}^{\circ}, b = a \sqcap a, \square A, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}}{\dfrac{b\in\breve{\mathbf{\Pi}}^{\circ} \mathbin{\scriptstyle\square} b = a \sqcap a, \square A, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}{\dfrac{(a \sqcap a)\in\breve{\mathbf{\Pi}}, \square A, a\in\breve{\mathbf{\Pi}}^{\circ} \Rightarrow [B/a]}{a\in\breve{\mathbf{\Pi}}^{\circ} \mathbin{\scriptstyle\square} (a \sqcap a)\in\breve{\mathbf{\Pi}}, \square A \Rightarrow [B/a]}}}}{a\in\breve{\mathbf{\Pi}}^{\circ}, \square A \Rightarrow [B/a]}{\scriptstyle +1}}{\dfrac{\square A \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ} \to [B/a]}{\square A \Rightarrow \square B}} \; .$$

Re 3.6x. Proof by induction on k. Essentially as for 3.6ix; left to the reader.

[18] I include this proof here because the one in [21], p. 1842, is flawed: some $\breve{\mathbf{\Pi}}$ should be $\breve{\mathbf{\Pi}}^{\circ}$.

Re 3.6xi. Employ 3.3ii:

$$\dfrac{\dfrac{\Rightarrow I \in \breve{\mathbf{\Pi}}^{\circ}}{\Rightarrow \Box(I \in \breve{\mathbf{\Pi}}^{\circ})}{+1} \qquad \dfrac{c \in \breve{\mathbf{\Pi}}^{\circ} \Rightarrow c^{I} \in \breve{\mathbf{\Pi}}^{\circ}}{\Box(c \in \breve{\mathbf{\Pi}}^{\circ}) \Rightarrow \Box(c^{I} \in \breve{\mathbf{\Pi}}^{\circ})}{+2} \qquad \Box(s \in \breve{\mathbf{\Pi}}^{\circ}), \Gamma \Rightarrow C}{s \in \breve{\mathbf{\Pi}}^{\circ}, \Gamma \Rightarrow C}{+1}\,.$$

QED

4. Successor and induction

DEFINITIONS 4.1. (1) $0 :\equiv \emptyset$.
(2) $s' :\equiv \{0, s\}$.

REMARKS 4.2. (1) Note that this successor notion is an 'exclusive' one, *i.e.*, one formulated in terms of identity.
(2) The definition of the successor of a term s along the line of $\{s, \{s\}\}$ doesn't lend itself to proving

$$s' = t' \to s = t$$

without induction. All that I was able to get is

$$s \in \mathbf{T} \sqcap t \in \mathbf{T} \sqcap s' = t' \to s = t\,.$$

This is why I adopt the above notion of the successor which allows the proof of 4.3vii without employing induction (and without employing any structural rules as shown in the next proposition).

PROPOSITION 4.3. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(4.3i) $s \in 0 \Rightarrow$;

(4.3ii) $\Rightarrow \{\{s\}\} \in s'$;

(4.3iii) $s' = 0 \Rightarrow$;

(4.3iv) $\{\{s\}\} \in t' \Rightarrow s \equiv t$;

(4.3v) $s' = t' \Rightarrow s \equiv t$;

(4.3vi) $s' = t' \Rightarrow s' \equiv t'$;

(4.3vii) $\mathfrak{F}[s], s' = t' \Rightarrow \mathfrak{F}[t]$.

Proof. Re 4.3i.

$$\dfrac{\bot \Rightarrow}{s \in \lambda \bot \Rightarrow}\,.$$

Re 4.3ii.

$$\dfrac{\dfrac{\Rightarrow \{\{s\}\} = \{\{s\}\}}{\Rightarrow \{\{s\}\} = \{\{0\},0\} \vee \{\{s\}\} = \{\{s\}\}}}{\Rightarrow \{\{s\}\}\in\{\{\{0\},0\},\{\{s\}\}\}}\;.$$

Re 4.3iii. Employ 4.3i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \{s\}\} \equiv \{\{s\}\}}{\Rightarrow \{\{s\}\} \equiv \{\{0\},0\} \vee \{\{s\}\} \equiv \{\{s\}\}}}{\Rightarrow \{\{s\}\}\in\{\{\{0\},0\},\{\{s\}\}\}} \qquad \{\{s\}\}\in 0 \Rightarrow}{\{\{s\}\}\in s' \to \{\{s\}\}\in 0 \Rightarrow}}{\{\{s\}\}\in s' \leftrightarrow \{\{s\}\}\in 0 \Rightarrow}}{\bigwedge x\,(x\in s' \leftrightarrow x\in 0) \Rightarrow}\;.$$

Re 4.3iv. Cf. [21], 127.35iv, p. 1745.
Re 4.3v. Employ 4.3ii and 4.3iv:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \{\{s\}\}\in s' \qquad \{\{s\}\}\in t' \Rightarrow s \equiv t}{\{\{s\}\}\in s' \to \{\{s\}\}\in t' \Rightarrow s \equiv t}}{\{\{s\}\}\in s' \leftrightarrow \{\{s\}\}\in t' \Rightarrow s \equiv t}}{s' = t' \Rightarrow s \equiv t}\;.$$

Re 4.3vi and 4.3vii. These are immediate consequences of 4.3v. QED

DEFINITION 4.4. $\mathbf{N}^{\circ} :\equiv \lambda x \bigwedge y(\bigwedge z(z\in y \to z'\in y) \supset (0\in y \to x\in y))$.[19]

REMARKS 4.5. (1) $\mathbf{N}^{\circ}$ is what I called an *exclusive* notion, *e.g.*, in [21], p. 1596, remark 116.6: it only contains the *numerals* $0, 0', 0'', \ldots$ and nothing else that may have the same numerical value but isn't really the same term, like, for instance, $0+0$.[20] This not only provides for the contractibility of wffs of the form $s\in\mathbf{N}^{\circ}$, but also for the possibility of proving a form of induction.

[19] Note the difference of the foregoing definition to that in [20], p. 400 (positioning of "step" and "basis": this is to get the "basis" from the left to the right side of weak implication).

[20] This is the difference to Frege and also Quine. Needless to say, that for them it is a confusion to make such a distinction and, thereby, to object to their conflation.

(2) As an immediate consequence of 4.3v, one has

$$s \in \mathbf{N}^\circ, s' = t' \Rightarrow t \in \mathbf{N}^\circ .$$

PROPOSITION 4.6. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(4.6i) $\Rightarrow 0 \in \mathbf{N}^\circ$;

(4.6ii) $s \in \mathbf{N}^\circ \Rightarrow s' \in \mathbf{N}^\circ$.

Proof. Pretty trivial and for that reason left to the reader; but the point to note is that 4.6ii does not require any **Z**-inference despite involving weak implication. This is a straightforward consequence of 2.16. QED

As in the case of proto-induction, I provide a list of schemata for derivable inferences together with an indication of how many **Z**-inferences go into it.

PROPOSITION 4.7. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_\lambda$*-derivable with an increase of the* **Z***-grade indicated on the right:*

(4.7i) $$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a']}{s \in \mathbf{N}^\circ, \Gamma \Rightarrow \mathfrak{F}[s]} {\scriptstyle +1} ;$$

(4.7ii) $$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a'] \qquad \mathfrak{F}[s], \Pi \Rightarrow C}{s \in \mathbf{N}^\circ, \Gamma, \Pi \Rightarrow C} {\scriptstyle +1} ;$$

(4.7iii) $$\frac{\Gamma \Rightarrow \mathfrak{F}[a, 0] \qquad \mathfrak{F}[s, b] \Rightarrow \mathfrak{F}[a, b']}{t \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[s, t]} {\scriptstyle +1} ;$$

(4.7iv) $$\frac{s \in \mathbf{N}^\circ, s \in \mathbf{N}^\circ, \Gamma \Rightarrow C}{s \in \mathbf{N}^\circ, \Gamma \Rightarrow C} {\scriptstyle +1} ;$$

(4.7v) $$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad a \in \mathbf{N}^\circ, \mathfrak{F}[a] \Rightarrow \mathfrak{F}[a']}{s \in \mathbf{N}^\circ, \Gamma \Rightarrow \mathfrak{F}[s]} {\scriptstyle +2} ;$$

(4.7vi) $$\frac{\Box(s \in \mathbf{N}^\circ), \Gamma \Rightarrow C}{s \in \mathbf{N}^\circ, \Gamma \Rightarrow C} {\scriptstyle +4} ;$$

(4.7vii) $$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \mathrm{k}[\mathfrak{F}[a]] \Rightarrow \mathfrak{F}[a']}{s \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[s]} {\scriptstyle +4} ;$$

(4.7viii)
$$\frac{s\in\mathbf{N}^\circ \Rightarrow \mathfrak{F}[s,0] \qquad \mathrm{k}[\mathfrak{F}[s,a]] \Rightarrow \mathfrak{F}[s,a'] \qquad \mathfrak{F}[s,t],\mathit{\Gamma}\Rightarrow C}{s\in\mathbf{N}^\circ, t\in\mathbf{N}^\circ,\mathit{\Gamma}\Rightarrow C}\,{\scriptstyle +9};$$

(4.7ix)
$$\frac{\mathit{\Gamma}\Rightarrow \mathfrak{F}[0,b] \qquad \Rightarrow \mathfrak{F}[a',0] \qquad \mathfrak{F}[a,b]\Rightarrow \mathfrak{F}[a',b']}{s\in\mathbf{N}, t\in\mathbf{B}\Rightarrow \mathfrak{F}[s,t]}\,{\scriptstyle +2}.$$

Proof. Re 4.7i. This is a straightforward consequence of the way $\mathbf{N}^\circ$ is defined, employing 3.6v:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{F}[a]\Rightarrow\mathfrak{F}[a']}{a\in\lambda x\,\mathfrak{F}[x]\Rightarrow a'\in\lambda x\,\mathfrak{F}[x]}}{\Rightarrow a\in\lambda x\,\mathfrak{F}[x]\to a'\in\lambda x\,\mathfrak{F}[x]}}{\Rightarrow \bigwedge z\,(z\in\lambda x\,\mathfrak{F}[x]\to z'\in\lambda x\,\mathfrak{F}[x])} \qquad \dfrac{\dfrac{\mathit{\Gamma}\Rightarrow\mathfrak{F}[0]}{\mathit{\Gamma}\Rightarrow 0\in\lambda x\,\mathfrak{F}[x]} \qquad \dfrac{\mathfrak{F}[s]\Rightarrow\mathfrak{F}[s]}{s\in\lambda x\,\mathfrak{F}[x]\Rightarrow\mathfrak{F}[s]}}{0\in\lambda x\,\mathfrak{F}[x]\to s\in\lambda x\,\mathfrak{F}[x],\mathit{\Gamma}\Rightarrow\mathfrak{F}[s]}}{\bigwedge z\,(z\in\lambda x\,\mathfrak{F}[x]\to z'\in\lambda x\,\mathfrak{F}[x])\supset(0\in\lambda x\,\mathfrak{F}[x]\to s\in\lambda x\,\mathfrak{F}[x]),\mathit{\Gamma}\Rightarrow\mathfrak{F}[s]}\,{\scriptstyle +1}}{\dfrac{\bigwedge y\,(\bigwedge z\,(z\in y\to z'\in y)\supset(0\in y\to s\in y)),\mathit{\Gamma}\Rightarrow\mathfrak{F}[s]}{s\in\lambda x\bigwedge y\,(\bigwedge z\,(z\in y\to z'\in y)\supset(0\in y\to x\in y)),\mathit{\Gamma}\Rightarrow\mathfrak{F}[s]}}\,.$$

Re 4.7ii. This is also a straightforward consequence of the way $\mathbf{N}^\circ$ is defined, employing 3.6v:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{F}[a]\Rightarrow\mathfrak{F}[a']}{a\in\lambda x\,\mathfrak{F}[x]\Rightarrow a'\in\lambda x\,\mathfrak{F}[x]}}{\Rightarrow a\in\lambda x\,\mathfrak{F}[x]\to a'\in\lambda x\,\mathfrak{F}[x]}}{\Rightarrow \bigwedge z\,(z\in\lambda x\,\mathfrak{F}[x]\to z'\in\lambda x\,\mathfrak{F}[x])} \qquad \dfrac{\dfrac{\mathit{\Gamma}\Rightarrow\mathfrak{F}[0]}{\mathit{\Gamma}\Rightarrow 0\in\lambda x\,\mathfrak{F}[x]} \qquad \dfrac{\mathfrak{F}[s],\mathit{\Pi}\Rightarrow C}{s\in\lambda x\,\mathfrak{F}[x],\mathit{\Pi}\Rightarrow C}}{0\in\lambda x\,\mathfrak{F}[x]\to s\in\lambda x\,\mathfrak{F}[x],\mathit{\Gamma},\mathit{\Pi}\Rightarrow C}}{\bigwedge z\,(z\in\lambda x\,\mathfrak{F}[x]\to z'\in\lambda x\,\mathfrak{F}[x])\supset(0\in\lambda x\,\mathfrak{F}[x]\to s\in\lambda x\,\mathfrak{F}[x]),\mathit{\Gamma},\mathit{\Pi}\Rightarrow C}\,{\scriptstyle +1}}{\dfrac{\bigwedge y\,(\bigwedge z\,(z\in y\to z'\in y)\supset(0\in y\to s\in y)),\mathit{\Gamma},\mathit{\Pi}\Rightarrow C}{s\in\lambda x\bigwedge y\,(\bigwedge z\,(z\in y\to z'\in y)\supset(0\in y\to x\in y)),\mathit{\Gamma},\mathit{\Pi}\Rightarrow C}}\,.$$

Re 4.7iii. Employ 4.7ii:

$$\dfrac{\dfrac{\mathit{\Gamma}\Rightarrow\mathfrak{F}[a,0]}{\mathit{\Gamma}\Rightarrow\bigwedge x\,\mathfrak{F}[x,0]} \qquad \dfrac{\dfrac{\mathfrak{F}[s,b]\Rightarrow\mathfrak{F}[a,b']}{\bigwedge x\,\mathfrak{F}[x,b]\Rightarrow\mathfrak{F}[a,b']}}{\bigwedge x\,\mathfrak{F}[x,b]\Rightarrow\bigwedge x\,\mathfrak{F}[x,b']} \qquad \dfrac{\mathfrak{F}[s,t]\Rightarrow\mathfrak{F}[s,t]}{\bigwedge x\,\mathfrak{F}[x,t]\Rightarrow\mathfrak{F}[s,t]}}{t\in\mathbf{N}^\circ,\mathit{\Gamma}\Rightarrow\mathfrak{F}[s,t]}\,{\scriptstyle +1}.$$

Re 4.7iv. As for 3.3iv only with 4.7ii instead of 3.3iii.
Re 4.7v. As for 3.3v only with 4.7iv and 4.7ii instead of 3.3v and 3.3iii.
Re 4.7vi. As for 3.6xi only with 4.7iv and 4.7ii instead of 3.3v and 3.3iii.
Re 4.7vii. Employ 3.6x, 4.7iv and 3.6v with an inference according to 4.7ii:

$$\dfrac{\dfrac{\Rightarrow \mathfrak{F}[0]}{\Rightarrow \Box\,\mathfrak{F}[0]}{\scriptstyle +1} \qquad \dfrac{\mathrm{k}[\mathfrak{F}[b]] \Rightarrow \mathfrak{F}[b']}{\Box\,\mathfrak{F}[b] \Rightarrow \Box\,\mathfrak{F}[b']}{\scriptstyle +2} \qquad \dfrac{\mathfrak{F}[t] \Rightarrow \mathfrak{F}[t]}{\Box\,\mathfrak{F}[t] \Rightarrow \mathfrak{F}[t]}}{t\in\mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[t]}{\scriptstyle +1}\,.$$

Re 4.7viii. Employ 3.6vii, 4.7vi, 3.6x and 4.7ii:

$$\dfrac{\dfrac{\dfrac{s\in\mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[s,0]}{\Box(s\in\mathbf{N}^{\circ}) \Rightarrow \Box\,\mathfrak{F}[s,0]}{\scriptstyle +2}}{s\in\mathbf{N}^{\circ} \Rightarrow \Box\,\mathfrak{F}[s,0]}{\scriptstyle +4} \qquad \dfrac{\mathrm{k}[\mathfrak{F}[s,a]] \Rightarrow \mathfrak{F}[s,a']}{\Box\,\mathfrak{F}[s,a] \Rightarrow \Box\,\mathfrak{F}[s,a']}{\scriptstyle +2} \qquad \dfrac{\mathfrak{F}[s,t], \varGamma \Rightarrow C}{\Box\,\mathfrak{F}[s,t], \varGamma \Rightarrow C}}{s\in\mathbf{N}^{\circ}, t\in\mathbf{N}^{\circ}, \varGamma \Rightarrow C}{\scriptstyle +1}\,.$$

Re 4.7ix. Let β be a fixed point: $\beta = \lambda x\,(x \equiv 0 \vee \bigvee y\,(y\in\beta \mathbin{\scriptstyle\Box} y' \equiv x))$. First, the following deduction is in $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mathfrak{F}[a',0]}{c\equiv 0 \Rightarrow \mathfrak{F}[a',c]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{F}[a,b] \Rightarrow \mathfrak{F}[a',b']}{b\in\beta, \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',b']}}{b\in\beta, b'\equiv c, \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',c]}}{b\in\beta \mathbin{\scriptstyle\Box} b'\equiv c, \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',c]}}{\bigvee y\,(y\in\beta \mathbin{\scriptstyle\Box} y'\equiv c), \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',c]}}{c\equiv 0 \vee \bigvee y\,(y\in\beta \mathbin{\scriptstyle\Box} y'\equiv c), \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',c]}}{c\in\beta, \bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \mathfrak{F}[a',c]}}{\bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow c\in\beta \to \mathfrak{F}[a',c]}}{\bigwedge y\,(y\in\beta \to \mathfrak{F}[a,y]) \Rightarrow \bigwedge y\,(y\in\beta \to \mathfrak{F}[a',y])}\,.$$

The next step is to establish the following, employing one **Z**-inference:

$$
\dfrac{\dfrac{\Rightarrow 0\in\beta \qquad \dfrac{\dfrac{\dfrac{\dfrac{a\in\beta \Rightarrow a\in\beta \qquad \Rightarrow a' \equiv a'}{a\in\beta \Rightarrow a\in\beta \,\Box\, a' \equiv a'}}{a\in\beta \Rightarrow \bigvee y\,(y\in\beta \,\Box\, y' \equiv a')}}{a\in\beta \Rightarrow a' \equiv 0 \vee \bigvee y\,(y\in\beta \,\Box\, y' \equiv a')}}{a\in\beta \Rightarrow a'\in\beta}}{t\in\mathbf{N}^{\circ} \Rightarrow t\in\beta}{\scriptstyle +1} \qquad \mathfrak{F}[s,t] \Rightarrow \mathfrak{F}[s,t]}{\dfrac{t\in\beta \rightarrow \mathfrak{F}[s,t], t\in\mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[s,t]}{\bigwedge y\,(y\in\beta \rightarrow \mathfrak{F}[s,y]), t\in\mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[s,t]\mathfrak{F}[s,y])}}\ .
$$

Finally an inference according to 4.7ii. Let $\bigwedge^{\beta} y\, \mathfrak{F}[y] :\equiv \bigwedge x\,(y\in\beta \rightarrow \mathfrak{F}[y])$ to save space:

$$
\dfrac{\dfrac{\mathit{\Gamma} \Rightarrow \mathfrak{F}[0,b]}{\mathit{\Gamma} \Rightarrow \bigwedge^{\beta} y\, \mathfrak{F}[0,y]} \quad \bigwedge^{\beta} y\, \mathfrak{F}[a,y] \Rightarrow \bigwedge^{\beta} y\, \mathfrak{F}[a',y] \quad \bigwedge^{\beta} y\, \mathfrak{F}[s,y], t\in\mathbf{N}^{\circ} \Rightarrow \mathfrak{F}[s,t]}{s\in\mathbf{N}, t\in\mathbf{N}^{\circ}, \mathit{\Gamma} \Rightarrow \mathfrak{F}[s,t]}{\scriptstyle +1}\ .
$$

QED

REMARKS 4.8. (1) 4.7i is a straightforward consequence of the way $\mathbf{N}^{\circ}$ is defined and is just the usual second order way of dealing with induction which is actually all that is needed in the classical case. Without contraction, however, this is not yet quite sufficient for proving the relevant properties of primitive recursive functions and this is why the further schemata are introduced.
(2) 4.7ii is designed to avoid cuts that would become necessary if 4.7i were employed in the case, *e.g.*, of the totality proofs below.
(3) 4.7iii is discussed in [13], p. 348, under the label *Erweiterung des Induktionsschemas* ("extension of the schema of induction") and is only listed here for the sake of interest.
(4) 4.7v deals with the situation that the induction step in turn depends on the free variable having only values in $\mathbf{N}^{\circ}$.
(5) 4.7vii deals with the situation that the "induction step" requires the induction hypothesis more than once. Suppose, for instance, that all we can get is

$$\Rightarrow \mathfrak{F}[0] \qquad \text{and} \qquad 2[\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a']\,.$$

Then, in order to get to $\Rightarrow \mathfrak{F}[3]$, we need the induction basis 2^3-times:

$$8[\mathfrak{F}[0]] \Rightarrow 4[\mathfrak{F}[1]]\,, \quad 4[\mathfrak{F}[1]] \Rightarrow 2[\mathfrak{F}[2]]\,, \quad 2[\mathfrak{F}[2]] \Rightarrow \mathfrak{F}[3]\,.$$

That's what is here being accounted for by the necessity notion:

$$\frac{\mathrm{k}[\mathfrak{F}[a]] \Rightarrow \mathfrak{F}[a']}{\Box\,\mathfrak{F}[a] \Rightarrow \Box\,\mathfrak{F}[a']}\,.$$

One might think of defining

$$\mathbf{N}^{2^\circ} :\equiv \lambda x \bigwedge y (\bigwedge z ([z \in y]^{\cdot 2} \to z' \in y) \supset (0 \in y \to x \in y))$$

in order to have an induction that tolerates two assumptions in the induction step. This would give

$$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \mathrm{k}[\mathfrak{F}[a]] \Rightarrow \mathfrak{F}[a']}{s \in [\mathbf{N}^\circ]^{\mathrm{k}} \Rightarrow \mathfrak{F}[s]}\,,$$

but not $s \in \mathbf{N}^{2^\circ} \Rightarrow s' \in \mathbf{N}^{2^\circ}$, which makes the whole thing useless.
(6) As 4.7ii, 4.7viii is designed to avoid cuts that would become necessary if 4.7i were employed.
(7) 4.7ix is a "double induction" without "nesting". My reason to include it here is that, unlike "nested double induction", it is perfectly $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$-derivable. The following "nested schema of induction" (without side wffs) is discussed in [13], p. 352 (*verschränktes Induktionsschema*):

$$\frac{\Rightarrow \mathfrak{F}[0,b] \qquad \mathfrak{F}[a,t_1] \Rightarrow \mathfrak{F}[a',0] \qquad \mathfrak{F}[a,t_2], \mathfrak{F}[a',b] \Rightarrow \mathfrak{F}[a',b']}{s \in \mathbf{N}, t \in \mathbf{N}, \Gamma, \Pi, \Xi \Rightarrow \mathfrak{F}[s,t]}\,.$$

This does not only require two inductions, but also a side wff in the first induction which, in the dialectical case, can only be accommodated for by introducing a necessity operator $\Box$ which spoils the classical reduction:

$$\dfrac{\dfrac{\Rightarrow \mathfrak{F}[0,b]}{\Rightarrow \bigwedge y\,\mathfrak{F}[0,b]} \qquad \dfrac{\dfrac{\dfrac{\mathfrak{F}[a,t_1] \Rightarrow \mathfrak{F}[a',0]}{\bigwedge y\,\mathfrak{F}[a,y] \Rightarrow \mathfrak{F}[a',0]} \qquad \dfrac{\mathfrak{F}[a,t_2], \mathfrak{F}[a',b] \Rightarrow \mathfrak{F}[a',b']}{\bigwedge y\,\mathfrak{F}[a,y], \mathfrak{F}[a',b] \Rightarrow \mathfrak{F}[a',b']}}{\bigwedge y\,\mathfrak{F}[a,y] \Rightarrow \mathfrak{F}[a',c]}}{\bigwedge y\,\mathfrak{F}[a,y] \Rightarrow \bigwedge y\,\mathfrak{F}[a',y]}}{\dfrac{\Rightarrow \bigwedge y\,\mathfrak{F}[s,y]}{\Rightarrow \mathfrak{F}[s,t]}}\,.$$

How this can be treated will be the topic of another paper, following the approach of my [23], section 5, pp. 136–159.

Proposition 4.9. $\mathbf{L^{i}D}_{\lambda}^{\mathrm{Z}\restriction_1} \vdash\ s'\in\mathbf{N}^{\circ} \Rightarrow s\in\mathbf{N}^{\circ}$.

Proof. Re 4.9. This is the reversal of 4.6ii; it requires a **Z**-inference. Let $\mathfrak{N} :\equiv 0 \equiv *_1 \vee \bigvee y(y\in\mathbf{N}^{\circ}\square\, y' \equiv *_1)$ and show $\Rightarrow \mathfrak{N}[0]$, $\mathfrak{N}[c] \Rightarrow \mathfrak{N}[c']$, and $\mathfrak{N}[s'] \Rightarrow s\in\mathbf{N}^{\circ}$:

$$\dfrac{\Rightarrow 0\equiv 0}{\Rightarrow 0\equiv 0\vee\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv 0)}\ .$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0'\in\mathbf{N}^{\circ}\qquad c\equiv 0\Rightarrow 0'\equiv c'}{c\equiv 0\Rightarrow 0'\in\mathbf{N}^{\circ}\square\, 0'\equiv c'}}{c\equiv 0\Rightarrow\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}\quad \dfrac{\dfrac{\dfrac{b\in\mathbf{N}^{\circ}, b'\equiv c\Rightarrow b'\in\mathbf{N}^{\circ}\square\, b''\equiv c'}{b\in\mathbf{N}^{\circ}, b'\equiv c\Rightarrow\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}}{b\in\mathbf{N}^{\circ}\square\, b'\equiv c\Rightarrow\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}}{\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c)\Rightarrow\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}}{c\equiv 0\vee\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c)\Rightarrow\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}}{c\equiv 0\vee\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c)\Rightarrow c'\equiv 0\vee(\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv c')}\ .$$

Now employ 4.3iii. and 4.3v:

$$\dfrac{s'\equiv 0\Rightarrow\qquad \dfrac{\dfrac{b\in\mathbf{N}^{\circ}, b'\equiv s'\Rightarrow s\in\mathbf{N}^{\circ}}{b\in\mathbf{N}^{\circ}\square\, b'\equiv s'\Rightarrow s\in\mathbf{N}^{\circ}}}{\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv s')\Rightarrow s\in\mathbf{N}^{\circ}}}{s'\equiv 0\vee\bigvee y(y\in\mathbf{N}^{\circ}\square\, y'\equiv s')\Rightarrow s\in\mathbf{N}^{\circ}}\ .$$

QED

Remark 4.10. The successor operation from definition 4.1 (2) can be turned into a successor *function* more in tune with the other definitions of functions that are to come:

$$\mathbf{S} :\equiv \lambda x_1 x_2 (x_2 = x_1')\ .$$

Obviously, $\mathbf{S}[\![t]\!] = t'$ would then be $\mathbf{L^{i}D}_{\lambda}$-deducible. Due to the exclusive character of $\mathbf{N}^{\circ}$, however, $s\in\mathbf{N}^{\circ} \to \mathbf{S}[\![s]\!]\in\mathbf{N}^{\circ}$ would not hold, only $s\in\mathbf{N} \to \mathbf{S}[\![s]\!]\in\mathbf{N}$, where $\mathbf{N}$ is defined as in 5.3 below.

5. Predecessor

The predecessor function can be defined without employing the fixed point property.

Definition 5.1. $pd :\equiv \lambda xy((x = 0 \wedge y = 0) \vee \bigvee z(z' = x \,\square\, z = y))$.

PROPOSITION 5.2. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(5.2i) $\Rightarrow pd[\![0]\!] = 0\,;$

(5.2ii) $a \in s, \langle s', b\rangle \in pd \Rightarrow a \in b\,;$

(5.2iii) $\Rightarrow \langle s', s\rangle\,;$

(5.2iv) $\Rightarrow pd[\![s']\!] = s\,.$

Proof. Re 5.2i

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0 = 0 \qquad \Rightarrow 0 = 0}{\Rightarrow 0 = 0 \wedge 0 = 0}}{\Rightarrow (0 = 0 \wedge 0 = 0) \vee \bigvee z\,(z' = 0 \,\square\, z = 0)}}{\Rightarrow \langle 0, 0\rangle \in pd} \qquad a \in 0 \Rightarrow a \in s}{\langle 0, 0\rangle \in pd \to a \in 0 \Rightarrow a \in s}}{\bigwedge y\,(\langle 0, y\rangle \in pd \to a \in y) \Rightarrow a \in s}}{a \in pd[\![0]\!] \Rightarrow a \in 0} \qquad a \in 0 \Rightarrow a \in pd[\![0]\!]}{\Rightarrow pd[\![0]\!] = 0}\;.$$

Re 5.2ii. Employ 4.3ii and 4.3i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \{\{s\}\} \in s' \qquad \{\{s\}\} \in 0 \Rightarrow}{\{\{s\}\} \in s' \to \{\{s\}\} \in 0 \Rightarrow}}{s' = 0 \Rightarrow}}{s' = 0 \wedge b = 0 \Rightarrow} \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \in s, s = b \Rightarrow a \in b}{a \in s, c = s, c = b \Rightarrow a \in b}}{a \in s, c' = s', c = b \Rightarrow a \in b}}{a \in s, c' = s' \,\square\, c = b \Rightarrow a \in b}}{a \in s, \bigvee z\,(z' = s' \,\square\, z = b) \Rightarrow a \in b}}{a \in s, (s' = 0 \wedge b = 0) \vee \bigvee z\,(z' = s' \,\square\, z = b) \Rightarrow a \in b}}{a \in s, \langle s', b\rangle \in pd \Rightarrow a \in b}\;.$$

Re 5.2iii.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow s' = s' \qquad \Rightarrow s = s}{\Rightarrow s' = s' \,\square\, s = s}}{\Rightarrow \bigvee z\,(z' = s' \,\square\, z = s)}}{\Rightarrow (s' = 0 \wedge s = 0) \vee \bigvee z\,(z' = s' \,\square\, z = s)}}{\Rightarrow \langle s', s\rangle \in pd}\;.$$

Re 5.2iv. Employ 5.2iii and 5.2ii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \langle s', s\rangle \in pd \qquad a\in s \Rightarrow a\in s}{\langle s', s\rangle \in pd \rightarrow a\in s \Rightarrow a\in s}}{\bigwedge y\,(\langle s', y\rangle \in pd \rightarrow a\in y) \Rightarrow a\in s}}{a\in pd\llbracket s'\rrbracket \Rightarrow a\in s} \qquad \dfrac{\dfrac{\dfrac{a\in s, \langle s', b\rangle \in pd \Rightarrow a\in b}{a\in s \Rightarrow \langle s', b\rangle \in pd \rightarrow a\in b}}{a\in s \Rightarrow \bigwedge y\,(\langle s', y\rangle \in pd \rightarrow a\in y)}}{a\in s \Rightarrow a\in pd\llbracket s'\rrbracket}}{\Rightarrow pd\llbracket s'\rrbracket = s}\,.$$

QED

Next comes the totality of the predecessor function. It will be obvious that totality can't hold for the predecessor function in the sense that it does for the successor operation as established in 4.6i: $pd\llbracket s\rrbracket$ just isn't in $\mathbf{N}^{\circ}$, no matter what its numerical value.[21] So in order to be able to establish some sort of totality we will have to shift to an inclusive notion of natural numbers.

Definition 5.3. $\mathbf{N} :\equiv \lambda x \bigvee y\,(y\in\mathbf{N}^{\circ} \sqcap x = y)$.[22]

Proposition 5.4. *The following is* $\mathbf{L^iD}_{\lambda}$*-deducible:*

(5.4i) $\Rightarrow 0\in\mathbf{N}^{\circ} \wedge pd\llbracket 0\rrbracket \in\mathbf{N}$;

(5.4ii) $c\in\mathbf{N}^{\circ} \wedge pd\llbracket c\rrbracket \in\mathbf{N} \Rightarrow c'\in\mathbf{N}^{\circ} \wedge pd\llbracket c'\rrbracket \in\mathbf{N}$;

(5.4iii) $pd\llbracket a\rrbracket \in\mathbf{N} \wedge a\in\mathbf{N}^{\circ}, s = a \Rightarrow pd\llbracket s\rrbracket \in\mathbf{N}$.

Proof. Re 5.4i. Employ 5.2i:

$$\dfrac{\Rightarrow 0\in\mathbf{N}^{\circ} \qquad \dfrac{\dfrac{\dfrac{\Rightarrow 0\in\mathbf{N}^{\circ} \qquad \Rightarrow pd\llbracket 0\rrbracket = 0}{\Rightarrow 0\in\mathbf{N}^{\circ} \sqcap pd\llbracket 0\rrbracket = 0}}{\Rightarrow \bigvee y\,(y\in\mathbf{N}^{\circ} \sqcap pd\llbracket 0\rrbracket = y)}}{\Rightarrow pd\llbracket 0\rrbracket \in\mathbf{N}}}{\Rightarrow 0\in\mathbf{N}^{\circ} \wedge pd\llbracket 0\rrbracket \in\mathbf{N}}\,.$$

[21] Cf. remark 4.5 (1) above.

[22] In [21], p. 1881, definition 136.48, I introduced a notion of **N** that involved $\Box$ and in that way provided for more than just one substitution. This, however, is not needed in the present context and since it is likely to be the source of some increase of **Z**-inferences we may well stick to a more restricted notion — as actually in the case of $\breve{\mathbf{\Pi}}$.

Re 5.4ii. Employ 5.2iv:

$$\dfrac{\dfrac{c\in\mathbf{N}^\circ \Rightarrow c'\in\mathbf{N}^\circ \qquad \dfrac{\dfrac{\dfrac{c\in\mathbf{N}^\circ \Rightarrow c\in\mathbf{N}^\circ \quad \Rightarrow pd\llbracket c'\rrbracket = c}{c\in\mathbf{N}^\circ \Rightarrow c\in\mathbf{N}^\circ \,\square\, pd\llbracket c'\rrbracket = c}}{c\in\mathbf{N}^\circ \Rightarrow \bigvee y\,(y\in\mathbf{N}^\circ \,\square\, pd\llbracket c'\rrbracket = y)}}{c\in\mathbf{N}^\circ \Rightarrow pd\llbracket c'\rrbracket\in\mathbf{N}}}{c\in\mathbf{N}^\circ \Rightarrow c'\in\mathbf{N}^\circ \wedge pd\llbracket c'\rrbracket\in\mathbf{N}}}{c\in\mathbf{N}^\circ \wedge pd\llbracket c\rrbracket\in\mathbf{N} \Rightarrow c'\in\mathbf{N}^\circ \wedge pd\llbracket c'\rrbracket\in\mathbf{N}}\ .$$

Re 5.4iii.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b\in\mathbf{N}^\circ, pd\llbracket s\rrbracket = b \Rightarrow b\in\mathbf{N}^\circ \,\square\, pd\llbracket s\rrbracket = b}{b\in\mathbf{N}^\circ, pd\llbracket s\rrbracket = b \Rightarrow \bigvee y\,(y\in\mathbf{N}^\circ \,\square\, pd\llbracket s\rrbracket = y)}}{b\in\mathbf{N}^\circ, pd\llbracket s\rrbracket = b \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{b\in\mathbf{N}^\circ, pd\llbracket a\rrbracket = b, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{b\in\mathbf{N}^\circ \,\square\, pd\llbracket a\rrbracket = b, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{\bigvee y\,(y\in\mathbf{N}^\circ \,\square\, pd\llbracket a\rrbracket = y), s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{pd\llbracket a\rrbracket\in\mathbf{N}, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{a\in\mathbf{N}^\circ \wedge pd\llbracket a\rrbracket\in\mathbf{N}, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}\ .$$

QED

PROPOSITION 5.5. $\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction_1} \vdash\ s\in\mathbf{N} \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}$.

Proof. Employ 5.4i–5.4iii with an induction inference according to 4.7ii and continue as follows:

$$\dfrac{\dfrac{a\in\mathbf{N}^\circ, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}{a\in\mathbf{N}^\circ \,\square\, s = a \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}}{\bigvee y\,(y\in\mathbf{N}^\circ \,\square\, s = y) \Rightarrow pd\llbracket s\rrbracket\in\mathbf{N}}\ .$$

QED

6. Recursion equations for addition

PROPOSITION 6.1. *There exists a term $\mathcal{A}$ such that:*

$$\mathbf{L^iD}_\lambda \vdash \mathcal{A} = \lambda x_1 x_2 x_3\,((x_2 = 0 \wedge x_3 = x_1) \vee$$
$$\bigvee y_1 \bigvee y_2 \bigvee y_3\,(y_1 = x_1 \,\square\, y_2' = x_2 \,\square\, y_3' = x_3 \,\square\, \langle\!\langle\langle\!\langle y_1, y_2\rangle\!\rangle, y_3\rangle\!\rangle\in\mathcal{A})\ .$$

Proof. This is an immediate consequence of the fixed-point property. QED

CONVENTIONS 6.2. (1) The following abbreviation is introduced to simplify presentation:

$$\mathcal{A}_s :\equiv \lambda x_2 x_3 \, (x_2 = 0 \wedge x_3 = s) \vee \\ \bigvee y_1 \bigvee y_2 \, (y_1' = x_2 \Box y_2' = x_3 \Box \langle\langle s, y_2\rangle, y_3\rangle \in \mathcal{A}) \, .$$

The full definition is only really needed in the proof of 6.7ii below.
(2) In order to save space in presentations, I shall occasionally use the following abbreviations:

$bas_{\mathcal{A}_s}[t, r]$ for $t = 0 \wedge r = s$, and
$stp_{\mathcal{A}_s}[t, r]$ for $\bigvee y_1 \bigvee y_2 \, (y_1' = t \Box y_2' = r \Box \langle\langle s, y_1\rangle, y_2\rangle \in \mathcal{A})$.

PROPOSITION 6.3. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.3i) $bas_{\mathcal{A}_s}[s', r] \Rightarrow$;

(6.3ii) $stp_{\mathcal{A}_s}[0, r] \Rightarrow$.

Proof. *Re* 6.3i.

$$\dfrac{\dfrac{s' = 0 \Rightarrow}{s' = 0 \wedge r = s \Rightarrow}}{bas_{\mathcal{A}_s}[s', r] \Rightarrow} \, .$$

Re 6.3ii.

$$\dfrac{\dfrac{\dfrac{c_1' = 0 \Rightarrow}{c_1' = 0, c_2' = r, \langle c_1, c_2\rangle \in \mathcal{A}_s \Rightarrow}}{c_1' = 0 \Box c_2' = r \Box \langle c_1, c_2\rangle \in \mathcal{A}_s \Rightarrow}}{\bigvee y_1 \bigvee y_2 \, (y_1' = 0 \Box y_2' = r \Box \langle y_1, y_2\rangle \in \mathcal{A}_s) \Rightarrow} \, .$$

QED

PROPOSITION 6.4. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.4i) $s_1 = s_2, t_1 = t_2, r_1 = r_2, \langle\langle s_1, t_1\rangle, r_1\rangle \in \mathcal{A} \Rightarrow \langle\langle s_2, t_2\rangle, r_2\rangle \in \mathcal{A}$;

(6.4ii) $\langle t, r\rangle \in \mathcal{A}_s \Leftrightarrow \langle\langle s, t\rangle, r\rangle \in \mathcal{A}$.

Proof. This is a straightforward consequence of the way $\mathcal{A}$ and $\mathcal{A}_s$ are defined. QED

Proposition 6.5. *Inferences according to the following schema are* $\mathbf{L^iD}_\lambda$-*derivable:*

$$\dfrac{c' = r, \langle t, c\rangle \in \mathcal{A}_s, \varGamma \Rightarrow C}{stp_{\mathcal{A}_s}[t', r], \varGamma \Rightarrow C}\ .$$

Proof. Employ 6.4i:

$$\begin{array}{c} c' = r, \langle\langle s, t\rangle, c\rangle \in \mathcal{A}, \varGamma \Rightarrow C \\ \hline b = t, c' = r, \langle\langle s, b\rangle, c\rangle \in \mathcal{A}, \varGamma \Rightarrow C \\ \hline b' = t', c' = r, \langle\langle s, b\rangle, c\rangle \in \mathcal{A}, \varGamma \Rightarrow C \\ \hline\hline b' = t' \square c' = r \square \langle\langle s, b\rangle, c\rangle \in \mathcal{A}, \varGamma \Rightarrow C \\ \hline\hline \bigvee y_1 \bigvee y_2\, (y_1' = t' \square y_2' = r \square \langle\langle s, y_1\rangle, y_2\rangle \in \mathcal{A}), \varGamma \Rightarrow C \end{array}\ .$$

QED

Definition 6.6. $s{+}t :\equiv \mathcal{A}_s[\![t]\!]$. I shall use $\mathcal{A}_s[\![t]\!]$ and $s{+}t$ interchangeably.

The first thing to establish about this definition is that it is substitutionally transparent.

Proposition 6.7. *The following is* $\mathbf{L^iD}_\lambda$-*deducible:*

(6.7i) $\quad s = t \Rightarrow r + s = r + t\,;$

(6.7ii) $\quad s = t \Rightarrow s + r = t + r\,.$

Proof. *Re* 6.7i.

$$\begin{array}{c} \dfrac{s = t, t = a_1' \Rightarrow s = a_1' \qquad \dfrac{b = a_2' \Rightarrow b = a_2' \qquad \langle a_1, a_2\rangle \in \mathcal{A}_r \Rightarrow \langle a_1, a_2\rangle \in \mathcal{A}_r}{b = a_2', \langle a_1, a_2\rangle \in \mathcal{A}_r \Rightarrow b = a_2' \square \langle a_1, a_2\rangle \in \mathcal{A}_r}}{s = t, t = a_1', b = a_2', \langle a_1, a_2\rangle \in \mathcal{A}_r \Rightarrow s = a_1' \square b = a_2' \square \langle a_1, a_2\rangle \in \mathcal{A}_r} \\ \hline s = t, t = a_1', b = a_2', \langle a_1, a_2\rangle \in \mathcal{A}_r \Rightarrow stp_{\mathcal{A}_r}[s, b] \\ \hline\hline s = t, t = a_1' \square b = a_2' \square \langle a_1, a_2\rangle \in \mathcal{A}_r \Rightarrow stp_{\mathcal{A}_r}[s, b] \\ \hline s = t, stp_{\mathcal{A}_r}[t, b] \Rightarrow stp_{\mathcal{A}_r}[s, b] \end{array}\ ;$$

$$\dfrac{\dfrac{\dfrac{\dfrac{s = t, bas_{\mathcal{A}_r}[t, b] \Rightarrow bas_{\mathcal{A}_r}[s, b]}{s = t, bas_{\mathcal{A}_r}[t, b] \Rightarrow bas_{\mathcal{A}_r}[s, b] \vee stp_{\mathcal{A}_r}[s, b]}}{s = t, bas_{\mathcal{A}_r}[t, b] \Rightarrow \langle s, b\rangle \in \mathcal{A}_r} \qquad \dfrac{\dfrac{s = t, stp_{\mathcal{A}_r}[t, b] \Rightarrow stp_{\mathcal{A}_r}[s, b]}{s = t, stp_{\mathcal{A}_r}[t, b] \Rightarrow bas_{\mathcal{A}_r}[s, b] \vee stp_{\mathcal{A}_r}[s, b]}}{s = t, stp_{\mathcal{A}_r}[t, b] \Rightarrow \langle s, b\rangle \in \mathcal{A}_r}}{s = t, bas_{\mathcal{A}_t}[s, b] \vee stp_{\mathcal{A}_r}[t, b] \Rightarrow \langle s, b\rangle \in \mathcal{A}_r}}{s = t, \langle t, b\rangle \in \mathcal{A}_r \Rightarrow \langle s, b\rangle \in \mathcal{A}_r}\ .$$

Re 6.7ii. First:

$$
\begin{array}{c}
r = 0 \Rightarrow r = 0 \qquad s = t, b = t \Rightarrow b = s \\
\hline
s = t, r = 0, b = t \Rightarrow r = 0 \mathbin{\square} b = s \\
\hline
s = t, r = 0, b = t \Rightarrow (r = 0 \mathbin{\square} b = s) \vee stp_{\mathcal{A}}[s, r, b] \\
\hline
s = t, r = 0, b = t \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \\
\hline
s = t,\, bas_{\mathcal{A}}[t, r, b] \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A}
\end{array} \; .
$$

Next:

$$
\begin{array}{c}
s = a_1 \Rightarrow s = a_1 \quad r = a_2' \Rightarrow s = a_1 \quad b = a_3' \Rightarrow b = a_3' \\
\hline
s = a_1, r = a_2', b = a_3' \Rightarrow s = a_1 \mathbin{\square} r = a_2' \mathbin{\square} b = a_3' \quad \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \\
\hline
s = a_1, r = a_2', b = a_3', \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow s = a_1 \mathbin{\square} r = a_2' \mathbin{\square} b = a_3' \mathbin{\square} \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \\
\hline
s = a_1, r = a_2', b = a_3', \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow stp_{\mathcal{A}}[s, r, b] \\
\hline
s = t, t = a_1, r = a_2', b = a_3', \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow stp_{\mathcal{A}}[s, r, b] \\
\hline
s = t, t = a_1, r = a_2', b = a_3', \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow bas_{\mathcal{A}}[s, r, b] \vee stp_{\mathcal{A}}[s, r, b] \\
\hline
s = t, t = a_1, r = a_2', b = a_3', \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \\
\hline\hline
s = t, t = a_1 \mathbin{\square} r = a_2' \mathbin{\square} b = a_3' \mathbin{\square} \langle\langle a_1, a_2\rangle, a_3\rangle \in \mathcal{A} \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \\
\hline
s = t, \bigvee y_1 \bigvee y_2 \bigvee y_3 \,(t = y_1 \mathbin{\square} r = y_2' \mathbin{\square} b = y_3' \mathbin{\square} \langle\langle y_1, y_2\rangle, y_3\rangle \in \mathcal{A}) \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A}
\end{array} \; .
$$

Finish as follows:

$$
\begin{array}{c}
s = t,\, bas_{\mathcal{A}}[t, r, b] \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \qquad s = t,\, stp_{\mathcal{A}}[t, r, b] \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \\
\hline
s = t,\, bas_{\mathcal{A}}[t, r, b] \vee stp_{\mathcal{A}}[t, r, b] \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A} \\
\hline\hline
s = t, \langle\langle t, r\rangle, b\rangle \in \mathcal{A} \Rightarrow \langle\langle s, r\rangle, b\rangle \in \mathcal{A}
\end{array} \; .
$$

QED

Proposition 6.8. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.8i) $\quad \Rightarrow \langle 0, s\rangle \in \mathcal{A}_s$;

(6.8ii) $\quad \langle t, r\rangle \in \mathcal{A}_s \Rightarrow \langle t', r'\rangle \in \mathcal{A}_s$.

Proof. *Re* 6.8i. Almost trivial; left to the reader.

Re 6.8ii.

$$\begin{array}{c}
\Rightarrow t' = t' \qquad \Rightarrow r' = r' \qquad \langle t,r\rangle\in\mathcal{A}_s \Rightarrow \langle t,r\rangle\in\mathcal{A}_s \\ \hline\hline
\langle t,r\rangle\in\mathcal{A}_s \Rightarrow t' = t' \square r' = r' \square \langle t,r\rangle\in\mathcal{A}_s \\ \hline\hline
\langle t,r\rangle\in\mathcal{A}_s \Rightarrow \bigvee y_1 \bigvee y_2\, (y_1' = t' \square y_2' = r' \square \langle y_1,y_2\rangle\in\mathcal{A}_s) \\ \hline
\langle t,r\rangle\in\mathcal{A}_s \Rightarrow (t' = 0 \square r' = s) \vee \bigvee y_1 \bigvee y_2\, (y_1' = t' \square y_2' = r' \square \langle y_1,y_2\rangle\in\mathcal{A}_s) \\ \hline
\langle t,r\rangle\in\mathcal{A}_s \Rightarrow \langle t',r'\rangle\in\mathcal{A}_s
\end{array} .$$

QED

Proposition 6.9. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.9i) $\Rightarrow s + 0 = s\,;$

(6.9ii) $\Rightarrow \langle 0, s+0\rangle\in\mathcal{A}_s\,.$

Proof. *Re* 6.9i. Employ 6.8i and 6.3ii:[23]

$$\begin{array}{c}
\begin{array}[b]{c}
\Rightarrow \langle 0,s\rangle\in\mathcal{A}_s \qquad a\in s \Rightarrow a\in s \\ \hline
\langle 0,s\rangle\in\mathcal{A}_s \to a\in s \Rightarrow a\in s \\ \hline
\bigwedge y\,(\langle 0,y\rangle\in\mathcal{A}_s \to a\in y) \Rightarrow a\in s \\ \hline
a\in(s+0) \Rightarrow a\in s
\end{array}
\qquad
\begin{array}[b]{c}
\begin{array}[b]{c}
a\in s, s = b \Rightarrow a\in b \\ \hline
a\in s, 0 = 0 \wedge s = b \Rightarrow a\in b
\end{array}
\qquad stp_{\mathcal{A}_s}[0,b] \Rightarrow \\ \hline
a\in s, bas_{\mathcal{A}_s}[0,b] \vee stp_{\mathcal{A}_s}[0,b] \Rightarrow a\in b \\ \hline
a\in s, \langle 0,b\rangle\in\mathcal{A}_s \Rightarrow a\in b \\ \hline
a\in s \Rightarrow \langle 0,b\rangle\in\mathcal{A}_s \to a\in b \\ \hline
a\in s \Rightarrow \bigwedge y\,(\langle 0,y\rangle\in\mathcal{A}_s \to a\in y) \\ \hline
a\in s \Rightarrow a\in(s+0)
\end{array} \\ \hline
\Rightarrow s + 0 = s
\end{array} .$$

Re 6.9ii. Employ 6.9i:

$$\begin{array}{c}
\Rightarrow 0 = 0 \qquad \Rightarrow s = s + 0 \\ \hline
\Rightarrow 0 = 0 \wedge s = s + 0 \\ \hline
\Rightarrow (0 = 0 \wedge s = s + 0) \vee \bigvee y_1 \bigvee y_2\, (y_1' = 0 \square y_2' = b \square \langle y_1,y_2\rangle\in\mathcal{A}_s) \\ \hline
\Rightarrow \langle 0, s+0\rangle\in\mathcal{A}_s
\end{array} .$$

QED

[23] Note that due to the fixed point definition of addition no **Z**-inference is needed here, in contrast to the classical approach as pursued in [21], p. 1889: proposition 137.13 requires an inference according to proposition 137.10 on p. 1887, and thereby a **Z**-inference.

PROPOSITION 6.10. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.10i) $\langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t') \Rightarrow a \in (s+t)'$;

(6.10ii) $uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)' \Rightarrow a \in (s+t')$;

(6.10iii) $uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow s+t' = (s+t)'$;

(6.10iv) $uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow \langle t', s+t'\rangle \in \mathcal{A}_s$.

Proof. Re 6.10i. As usual, this direction is almost trivial in view of how application is defined. Employ 6.8ii:

$$\begin{array}{c}
\langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow \langle t, (s+t)'\rangle \in \mathcal{A}_s \qquad a \in (s+t)' \Rightarrow a \in (s+t)' \\
\hline
\langle t, s+t\rangle \in \mathcal{A}_s, \langle t, (s+t)'\rangle \in \mathcal{A}_s \rightarrow a \in (s+t)' \Rightarrow a \in (s+t)' \\
\hline
\langle t, s+t\rangle \in \mathcal{A}_s, \bigwedge y\,(\langle t, y\rangle \in \mathcal{A}_s \rightarrow a \in y) \Rightarrow a \in (s+t)' \\
\hline
\langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t') \Rightarrow a \in (s+t)'
\end{array} .$$

Re 6.10ii. This is the direction which requires uniqueness. Employ 6.11iv:

$$\begin{array}{c}
\langle t, s+t\rangle \in \mathcal{A}_s, \langle t, c_2\rangle \in \mathcal{A}_s \Rightarrow \langle t, s+t\rangle \in \mathcal{A}_s \,\square\, \langle t, c_2\rangle \in \mathcal{A}_s \quad s+t = c_2, a \in (s+t)' \Rightarrow a \in c_2' \\
\hline
\langle t, s+t\rangle \in \mathcal{A}_s \,\square\, \langle t, c_2\rangle \in \mathcal{A}_s \rightarrow s+t = c_2, \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', \langle t, c_2\rangle \in \mathcal{A}_s \Rightarrow a \in c_2' \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', \langle t, c_2\rangle \in \mathcal{A}_s \Rightarrow a \in c_2' \\
\hline\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', c_1 = t, c_2' = b, \langle c_1, c_2\rangle \in \mathcal{A}_s \Rightarrow a \in b \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', c_1' = t', c_2' = b, \langle c_1, c_2\rangle \in \mathcal{A}_s \Rightarrow a \in b \\
\hline\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', c_1' = t' \,\square\, c_2' = b \,\square\, \langle c_1, c_2\rangle \in \mathcal{A}_s \Rightarrow a \in b \\
\hline\hline
bas_{\mathcal{A}_s}[t', b] \Rightarrow \qquad uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', stp_{\mathcal{A}_s}[t', b] \Rightarrow a \in b \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', bas_{\mathcal{A}_s}[t', b] \vee stp_{\mathcal{A}_s}[t', b] \Rightarrow a \in b \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)', \langle t', b\rangle \in \mathcal{A}_s \Rightarrow a \in b \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)' \Rightarrow \langle t', b\rangle \in \mathcal{A}_s \rightarrow a \in b \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)' \Rightarrow \bigwedge y\,(\langle t', y\rangle \in \mathcal{A}_s \rightarrow a \in y) \\
\hline
uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)' \Rightarrow a \in (s+t')
\end{array} .$$

Re 6.10iii. Employ 6.10i and 6.10ii:

$$\frac{\langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t') \Rightarrow a \in (s+t)' \quad uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s, a \in (s+t)' \Rightarrow a \in (s+t')}{uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow s+t' = (s+t)'}\,.$$

Re 6.10iv. Employ 6.10iii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow t' = t' \quad \dfrac{uni[t, \mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow (s+t)' = s+t' \quad \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow \langle t, s+t\rangle \in \mathcal{A}_s}{uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow (s+t)' = s+t' \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s}}{uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow t' = t' \,\square\, (s+t)' = s+t' \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s}}{\overline{uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow \bigvee y_1 \bigvee y_2\, (y_1' = t' \,\square\, y_2' = s+t' \,\square\, \langle y_1, y_2\rangle \in \mathcal{A}_s)}}}{uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow bas_{\mathcal{A}_s}[t', s+t'] \vee stp_{\mathcal{A}_s}[t', s+t']}}{uni[t, \mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow \langle t', s+t'\rangle \in \mathcal{A}_s}\,.$$

QED

PROPOSITION 6.11. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(6.11i) $\langle 0, r\rangle \in \mathcal{A}_s \Rightarrow s = r\,;$

(6.11ii) $\Rightarrow uni[0, \mathcal{A}_s]\,;$

(6.11iii) $uni[t, \mathcal{A}_s], stp_{\mathcal{A}_s}[s, a', c_1], stp_{\mathcal{A}_s}[s, a', c_2] \Rightarrow c_1 = c_2\,;$

(6.11iv) $uni[t, \mathcal{A}_s] \Rightarrow uni[t', \mathcal{A}_s]\,.$

Proof. Re 6.11i. Almost trivial, but nevertheless .... Employ 6.3ii:

$$\dfrac{\dfrac{\dfrac{s = r \Rightarrow s = r}{0 = 0 \wedge s = r \Rightarrow s = r} \quad \bigvee y_1 \bigvee y_2 (y_1' = 0 \,\square\, y_2' = r \,\square\, \langle y_1, y_2\rangle \in \mathcal{A}_s) \Rightarrow s = r}{(0 = 0 \,\square\, r = s) \vee \bigvee y_1 \bigvee y_2 (y_1' = 0 \,\square\, y_2' = r \,\square\, \langle y_1, y_2\rangle \in \mathcal{A}_s) \Rightarrow s = r}}{\langle 0, r\rangle \in \mathcal{A}_s \Rightarrow s = r}\,.$$

Re 6.11ii. Variation of 6.11i; left to the reader.
Re 6.11iii. The essential step is that of the 'reversibility' of the successor relation, in the sense of 4.3vii, which is being applied twice in the following deduction. This is where the new definition of the successor with 2.2 above comes in:

$$\begin{array}{c}
\langle a,b_2\rangle\in\mathcal{A}_s,\langle a,b_1\rangle\in\mathcal{A}_s\Rightarrow\langle a,b_1\rangle\in\mathcal{A}_s\,\square\,\langle a,b_2\rangle\in\mathcal{A}_s \qquad b_1=b_2\Rightarrow b_2'=b_1' \\ \hline
\langle a,b_1\rangle\in\mathcal{A}_s\,\square\,\langle t,b_2\rangle\in\mathcal{A}_s\to b_1=b_2,\langle t,b_2\rangle\in\mathcal{A}_s,\langle t,b_1\rangle\in\mathcal{A}_s\Rightarrow b_2'=b_1' \\ \hline
\mathit{uni}[t,\mathcal{A}_s],\langle t,b_2\rangle\in\mathcal{A}_s,\langle t,b_1\rangle\in\mathcal{A}_s\Rightarrow b_2'=b_1' \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],\langle t,b_2\rangle\in\mathcal{A}_s,a_1=t,b_1'=c_2,\langle a_1,b_1\rangle\in\mathcal{A}_s\Rightarrow b_2'=c_2 \\ \hline
\mathit{uni}[t,\mathcal{A}_s],\langle t,b_2\rangle\in\mathcal{A}_s,a_1'=t',b_1'=c_2,\langle a_1,b_1\rangle\in\mathcal{A}_s\Rightarrow b_2'=c_2 \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],\langle t,b_2\rangle\in\mathcal{A}_s,\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow b_2'=c_2 \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],b_1=t,b_2'=c_1,\langle b_1,b_2\rangle\in\mathcal{A}_s,\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2 \\ \hline
\mathit{uni}[t,\mathcal{A}_s],b_1'=t',b_2'=c_1,\langle b_1,b_2\rangle\in\mathcal{A}_s,\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2 \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],b_1'=t'\,\square\,b_2'=c_1\,\square\,\langle b_1,b_2\rangle\in\mathcal{A}_s,\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2 \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],\mathit{stp}_{\mathcal{A}_s}[t',c_1],\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2
\end{array}\ .$$

Re 6.11iv. Employ 6.11iii:

$$\begin{array}{c}
\mathit{bas}_{\mathcal{A}_s}[t',c_1]\Rightarrow \quad
\begin{array}{c}
\mathit{bas}_{\mathcal{A}_s}[t',c_2]\Rightarrow \quad \mathit{uni}[t,\mathcal{A}_s],\mathit{stp}_{\mathcal{A}_s}[t',c_1],\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2 \\ \hline
\mathit{uni}[t,\mathcal{A}_s],\mathit{stp}_{\mathcal{A}_s}[t',c_1],\mathit{bas}_{\mathcal{A}_s}[t',c_2]\vee\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2
\end{array} \\ \hline
\mathit{uni}[t,\mathcal{A}_s],\mathit{bas}_{\mathcal{A}_s}[t',c_1]\vee\mathit{stp}_{\mathcal{A}_s}[t',c_1],\mathit{bas}_{\mathcal{A}_s}[t',c_2]\vee\mathit{stp}_{\mathcal{A}_s}[t',c_2]\Rightarrow c_1=c_2 \\ \hline\hline
\mathit{uni}[t,\mathcal{A}_s],\langle t',c_1\rangle\in\mathcal{A}_s,\langle t',c_2\rangle\in\mathcal{A}_s\Rightarrow c_1=c_2 \\ \hline
\mathit{uni}[t,\mathcal{A}_s]\Rightarrow\mathit{uni}[t',\mathcal{A}_s]
\end{array}\ .$$

QED

REMARK 6.12. Two separate inductions would now do the job; a first one (according to 4.7i) to yield $t\in\mathbf{N}^{\circ}\Rightarrow\mathit{uni}[s,t,\mathcal{A}]$:

1. $\Rightarrow\mathit{uni}[0,\mathcal{A}_s]$ 6.11ii
2. $\mathit{uni}[b,\mathcal{A}_s]\Rightarrow\mathit{uni}[b',\mathcal{A}_s]$ 6.11iv

and a second one (according to 4.7vii, because of the double occurrence of the antecedent formula) to yield $t\in\mathbf{N}^{\circ}\Rightarrow\langle\langle s,t\rangle,s+t]]\rangle\in\mathcal{A}$:

3. $\Rightarrow\langle 0,s+0\rangle\in\mathcal{A}_s$ 6.9ii
5. $b\in\mathbf{N}^{\circ},2[\langle b,s+b\rangle\in\mathcal{A}_s]\Rightarrow\langle b',s+b'\rangle\in\mathcal{A}$

where the last one is obtained from 6.10iv

$$\frac{b\in\mathbf{N}^{\circ}\Rightarrow\mathit{uni}[s,b,\mathcal{A}] \qquad \mathit{uni}[s,b,\mathcal{A}],2[\langle b,s+b\rangle\in\mathcal{A}_s]\Rightarrow\langle b',s+b'\rangle\in\mathcal{A}}{b\in\mathbf{N}^{\circ},2[\langle b,s+b\rangle\in\mathcal{A}_s]\Rightarrow\langle b',s+b'\rangle\in\mathcal{A}}\ \text{cut}\ .$$

Altogether, this is not the most economical way to obtain the recursion equations for addition and this is why I combine the two inductions into one which saves considerably on **Z**-inferences.

PROPOSITION 6.13. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(6.13i) $\Rightarrow uni[0,\mathcal{A}_s] \,\square\, \langle 0, s+0\rangle \in \mathcal{A}_s$;

(6.13ii) $2\,[uni[t,\mathcal{A}_s] \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow uni[t',\mathcal{A}_s] \,\square\, \langle t', s+t'\rangle \in \mathcal{A}_s$.

Proof. *Re* 6.13i. Employ 6.11ii and 6.9ii:

$$\dfrac{\Rightarrow uni[0,\mathcal{A}_s] \qquad \Rightarrow \langle 0, s+0\rangle \in \mathcal{A}_s}{\Rightarrow uni[0,\mathcal{A}_s] \,\square\, \langle 0, s+0\rangle \in \mathcal{A}_s}\ .$$

Re 6.13ii. Employ 6.11iv and 6.10iv:

$$\dfrac{\dfrac{uni[t,\mathcal{A}_s] \Rightarrow uni[t',\mathcal{A}_s] \qquad uni[t,\mathcal{A}_s], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow \langle t', s+t'\rangle \in \mathcal{A}_s}{2[uni[t,\mathcal{A}_s]], 2[\langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow uni[t',\mathcal{A}_s] \,\square\, \langle t', s+t'\rangle \in \mathcal{A}_s}}{2[uni[t,\mathcal{A}_s] \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s] \Rightarrow uni[t',\mathcal{A}_s] \,\square\, \langle t', s+t'\rangle \in \mathcal{A}_s}\ .$$

QED

Everything so far has been $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$-deducible. Now come the final steps, the ones that involve **Z**-inferences, be that in the form of a "modal" inference or an induction.

PROPOSITION 6.14. $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{Z\restriction 4} \vdash\ t \in \mathbf{N}^{\circ} \Rightarrow s+t' = (s+t)'$.

Proof. Employ 6.13i and 6.13ii with an inference according to schema 4.7vii. QED

REMARKS 6.15. (1) One last time I want to spell out an induction in terms of higher order logic, *i.e.*, $\mathbf{N}^{\circ}$. Employ 6.13ii, 6.13i and 6.10iii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{Z\restriction 2}\\ \vdots \\ b\in\xi \Rightarrow b'\in\xi\end{matrix}}{\Rightarrow b\in\xi \to b'\in\xi}}{\Rightarrow \bigwedge z(z\in\xi \to z'\in\xi)} \quad \dfrac{\begin{matrix}\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{Z\restriction 1}\\ \vdots \\ \Rightarrow 0\in\xi\end{matrix} \quad \dfrac{\dfrac{\dfrac{uni[t,\mathcal{A}_s], \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow s+t' = (s+t)'}{uni[t,\mathcal{A}_s] \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s \Rightarrow s+t' = (s+t)'}}{\Box(uni[t,\mathcal{A}_s] \,\square\, \langle t, s+t\rangle \in \mathcal{A}_s) \Rightarrow s+t' = (s+t)'}}{t\in\xi \Rightarrow s+t' = (s+t)'}}{0\in\xi \to t\in\xi \Rightarrow s+t' = (s+t)'}}{\dfrac{\bigwedge z((z\in\xi) \to (z'\in\xi)) \supset ((0\in\xi) \to (t\in\xi)) \Rightarrow s+t' = (s+t)'}{t\in\mathbf{N}^{\circ} \Rightarrow s+t' = (s+t)'}}\,{}^{+1}\ ,$$

where $\xi :\equiv \lambda x \, \Box \, (uni[x, \mathcal{A}_s] \,\Box\, \langle x, s + x\rangle \in \mathcal{A}_s)$.
(2) It doesn't need much to see that the foregoing approach to proving the recursion equations for addition can be extended to all primitive recursive functions, *i.e.*, the recursion equations for all primitive recursive function are provable in $\mathbf{L^iD}_\lambda^{Z\restriction_4}$. The approach to establishing the recursion equations for addition can be divided into four blocks:

$$(1) \qquad \Rightarrow \langle 0, s\rangle \in \mathcal{A}_s\,; \qquad \langle t, r\rangle \in \mathcal{A}_s \Rightarrow \langle t', r'\rangle \in \mathcal{A}_s\,.$$

$$(2) \qquad \Rightarrow uni[0, \mathcal{A}_s]\,; \qquad uni[t, \mathcal{A}_s] \Rightarrow uni[t', \mathcal{A}_s]\,.$$

$$(3) \qquad uni[t, \mathcal{A}_s], \langle t, \mathcal{A}_s[\![t]\!]\rangle \in \mathcal{A}_s \Rightarrow \mathcal{A}_s[\![t']\!] = \mathcal{A}_s[\![t]\!]'\,;$$

$$(4) \qquad \Rightarrow uni[0, \mathcal{A}_s[\![0]\!]]\,; \qquad uni[t, \mathcal{A}_s], 2\,[\langle t, \mathcal{A}_s[\![t]\!]\rangle \in \mathcal{A}_s] \Rightarrow \langle t', \mathcal{A}_s[\![t']\!]\rangle \in \mathcal{A}_s\,.$$

This approach fits to all primitive-recursive functions. If h_s is a one-place primitive recursive function defined in terms of another one-place function f and a two-place function g_s as the fixed point

$$h_s = \lambda x_1 x_2 \, (x_1 = 0 \wedge x_2 = f[\![s]\!]) \vee \bigvee y_1 \bigvee y_2 \, (y_1' = x_1 \,\Box\, g_s[\![s, y_2]\!] = x_2 \,\Box\, \langle y_1, y_2\rangle \in h_s)\,,$$

then the essential ingredients for obtaining the recursion equations are:

$$(1) \qquad \Rightarrow \langle 0, f[\![s]\!]\rangle \in h_s\,; \qquad \langle t, r\rangle \in h_s \Rightarrow \langle t', g_s[\![t, r]\!]\rangle \in h_s\,.$$

$$(2) \qquad \Rightarrow uni[0, h_s]\,; \qquad uni[t, h_s] \Rightarrow uni[t', h_s]\,.$$

$$(3) \qquad uni[t, h_s], \langle t, h_s[\![t]\!]\rangle \in h_s \Rightarrow h_s[\![t']\!] = g_s[\![t, h_s[\![t]\!]]\!]\,.$$

$$(4) \qquad \Rightarrow uni[0, h_s[\![0]\!]]\,; \qquad uni[t, h_s], 2\,[\langle t, h_s[\![t]\!]\rangle \in h_s] \Rightarrow \langle t', h_s[\![t']\!]\rangle \in h_s\,.$$

I leave it at these hints trusting that they are sufficient to support my claim that the approach extends to all primitive-recursive functions.

7. Totality of addition

As with the predecessor function, totality can't hold for addition in the sense that it does for the successor operation: $s+t$ just won't be in $\mathbf{N}^\circ$, no matter what its numerical value. But that's what the notion of $\mathbf{N}$ (definition 5.3 above) has been introduced for: if $s\in\mathbf{N}$ and $t\in\mathbf{N}$ then $(s+t)\in\mathbf{N}$.

REMARK 7.1. It should be clear that the totality of addition can be established on the basis of the recursion equations as obtained in 6.9i and 6.14 above employing just another simple induction:

$$\dfrac{\dfrac{\dfrac{s\in\mathbf{N}^\circ \Rightarrow s\in\mathbf{N}^\circ \qquad \Rightarrow s = s+0}{s\in\mathbf{N}^\circ \Rightarrow s\in\mathbf{N}^\circ \,\square\, s = s+0}}{s\in\mathbf{N}^\circ \Rightarrow (s+0)\in\mathbf{N}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{c\in\mathbf{N}^\circ \Rightarrow c'\in\mathbf{N}^\circ \qquad \dfrac{b\in\mathbf{N}^\circ \Rightarrow (s+b)' = s+b'}{b\in\mathbf{N}^\circ, c = s+b \Rightarrow c' = s+b'}}{b\in\mathbf{N}^\circ, c\in\mathbf{N}^\circ, c = s+b \Rightarrow c'\in\mathbf{N}^\circ \,\square\, c' = s+b'}}{b\in\mathbf{N}^\circ, c\in\mathbf{N}^\circ \,\square\, c = s+b \Rightarrow c'\in\mathbf{N}^\circ \,\square\, c' = s+b'}}{b\in\mathbf{N}^\circ, c\in\mathbf{N}^\circ \,\square\, c = s+b \Rightarrow (s+b')\in\mathbf{N}}}{b\in\mathbf{N}^\circ, (s+b)\in\mathbf{N} \Rightarrow (s+b')\in\mathbf{N}}}{s\in\mathbf{N}^\circ, t\in\mathbf{N}^\circ \Rightarrow (s+t)\in\mathbf{N}}\ .$$

And then use this for a cut in the inference marked † below:

$$\dfrac{\dfrac{\dfrac{s\in\mathbf{N}^\circ, t\in\mathbf{N}^\circ \Rightarrow (s+t)\in\mathbf{N} \qquad \dfrac{\dfrac{\dfrac{\dfrac{c\in\mathbf{N}^\circ \Rightarrow c\in\mathbf{N}^\circ \qquad \dfrac{c = a+b \Rightarrow c = a+b}{a = s, b = t, c = s+b \Rightarrow c = s+t}}{c\in\mathbf{N}^\circ, a = s, b = t, c = s+b \Rightarrow c\in\mathbf{N}^\circ \,\square\, c = s+t}}{a = s, b = t, c\in\mathbf{N}^\circ \,\square\, c = s+b \Rightarrow c\in\mathbf{N}^\circ \,\square\, c = s+t}}{a = s, b = t, c\in\mathbf{N}^\circ \,\square\, c = a+b \Rightarrow (s+t)\in\mathbf{N}}}{a = s, b = t, (a+b)\in\mathbf{N} \Rightarrow (s+t)\in\mathbf{N}}}{a\in\mathbf{N}^\circ, b\in\mathbf{N}^\circ, a = s, b = t \Rightarrow (s+t)\in\mathbf{N}}\,\dagger}{a\in\mathbf{N}^\circ \,\square\, a = s, b\in\mathbf{N}^\circ \,\square\, b = t \Rightarrow (s+t)\in\mathbf{N}}}{s\in\mathbf{N}, t\in\mathbf{N} \Rightarrow (s+t)\in\mathbf{N}}\ .$$

The point is to get around this cut and the additional induction.

Proposition 7.2. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(7.2i) $s\in\mathbf{N}^{\circ}\Rightarrow(s+0)\in\mathbf{N}\,;$

(7.2ii) $uni[t,\mathcal{A}_s],\langle t,\mathcal{A}_s[\![t]\!]\rangle\in\mathcal{A}_s,(s+t)\in\mathbf{N})\Rightarrow(s+t')\in\mathbf{N}\,.$

Proof. Re 7.2i. Employ 6.9i:

$$\begin{array}{c}
s\in\mathbf{N}^{\circ}\Rightarrow s\in\mathbf{N}^{\circ}\qquad\Rightarrow(s+0)=s\\
\hline
s\in\mathbf{N}^{\circ}\Rightarrow s\in\mathbf{N}^{\circ}\,\square\,(s+0)=s\\
\hline
s\in\mathbf{N}^{\circ}\Rightarrow\bigvee y\,(y\in\mathbf{N}^{\circ}\,\square\,(s+0)=y)\\
\hline
s\in\mathbf{N}^{\circ}\Rightarrow(s+0)\in\mathbf{N}
\end{array}\;.$$

Re 7.2ii. Employ 6.14:

$$\begin{array}{c}
\begin{array}{cc}
 & uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s\Rightarrow s+t'=(s+t)'\\
 \cline{2-2}
b\in\mathbf{N}^{\circ}\Rightarrow b'\in\mathbf{N}^{\circ} & uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,s+t=b\Rightarrow s+t'=b'
\end{array}\\
\hline
uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,b\in\mathbf{N}^{\circ},s+t=b\Rightarrow b'\in\mathbf{N}^{\circ}\,\square\,s+t'=b'\\
\hline
uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,b\in\mathbf{N}^{\circ},s+t=b\Rightarrow\bigvee y\,(y\in\mathbf{N}^{\circ}\,\square\,s+t'=y)\\
\hline
t\in\mathbf{N}^{\circ},b\in\mathbf{N}^{\circ},s+t=b\Rightarrow(s+t')\in\mathbf{N}\\
\hline
uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,b\in\mathbf{N}^{\circ}\,\square\,s+t=b\Rightarrow(s+t')\in\mathbf{N}\\
\hline
uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,\bigvee y\,(y\in\mathbf{N}^{\circ}\,\square\,s+t=y)\Rightarrow(s+t')\in\mathbf{N}\\
\hline
uni[t,\mathcal{A}_s],\langle t,s+t\rangle\in\mathcal{A}_s,(s+t)\in\mathbf{N}\Rightarrow(s+t')\in\mathbf{N}
\end{array}\;.$$

QED

Proposition 7.3. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(7.3i) $s\in\mathbf{N}^{\circ}\Rightarrow uni[0,\mathcal{A}_s]\,\square\,\langle 0,s+0\rangle\in\mathcal{A}_s\,\square\,(s+0)\in\mathbf{N}\,;$

(7.3ii) $3\,[uni[t,\mathcal{A}_s]\,\square\,\langle t,s+t\rangle\in\mathcal{A}_s\,\square\,(s+t)\in\mathbf{N}]\Rightarrow;$
$uni[t',\mathcal{A}_s]\,\square\,\langle t',s+t'\rangle\in\mathcal{A}_s\,\square\,(s+t')\in\mathbf{N}\,;$

(7.3iii) $uni[b,\mathcal{A}_a]\,\square\,\langle b,a+b\rangle\in\mathcal{A}_a\,\square\,(a+b)\in\mathbf{N}),s=a,t=b\Rightarrow$
$(s+t)\in\mathbf{N}\,.$

Proof. Re 7.3i. Conjunction of 6.13i and 7.2i.
Re 7.3ii. Conjunction of 6.13ii and 7.2ii.

Re 7.3iii. Employ 6.7i and 6.7ii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{c{\in}\mathbf{N}^{\circ} \Rightarrow c{\in}\mathbf{N}^{\circ} \qquad \dfrac{s = a, t = b \Rightarrow (s+t) = (a+b)}{s = a, t = b, (a+b) = c \Rightarrow (s+t) = c}}{c{\in}\mathbf{N}^{\circ}, s = a, t = b, (a+b) = c \Rightarrow c{\in}\mathbf{N}^{\circ} \sqcap (s+t) = c}}{c{\in}\mathbf{N}^{\circ}, s = a, t = b, (a+b) = c \Rightarrow \bigvee y\,(y{\in}\mathbf{N}^{\circ} \sqcap (s+t) = y)}}{c{\in}\mathbf{N}^{\circ}, s = a, t = b, (a+b) = c \Rightarrow (s+t){\in}\mathbf{N}}}{c{\in}\mathbf{N}^{\circ} \sqcap (a+b) = c, s = a, t = b \Rightarrow (s+t){\in}\mathbf{N}}}{\bigvee y\,(y{\in}\mathbf{N}^{\circ} \sqcap (a+b) = y), s = a, t = b \Rightarrow (s+t){\in}\mathbf{N}}}{(a+b){\in}\mathbf{N}, s = a, t = b \Rightarrow (s+t){\in}\mathbf{N}}}{\dfrac{uni[b, \mathcal{A}_a], \langle b, a+b\rangle{\in}\mathcal{A}_a, (a+b){\in}\mathbf{N}, s = a, t = b \Rightarrow (s+t){\in}\mathbf{N}}{uni[b, \mathcal{A}_a] \sqcap \langle b, a+b\rangle{\in}\mathcal{A}_a \sqcap (a+b){\in}\mathbf{N}, s = a, t = b \Rightarrow (s+t){\in}\mathbf{N}}} \, .$$

QED

PROPOSITION 7.4. $\mathbf{L^iD}_{\lambda}^{Z\restriction 9} \vdash \; s{\in}\mathbf{N}, t{\in}\mathbf{N} \Rightarrow (s+t){\in}\mathbf{N}\,.$

Proof. This is an immediate consequence of 7.3i, 7.3ii, and 7.3iii by means of 4.7viii. QED

8. Multiplication

The schema of the foregoing two sections will now be applied to multiplication. In view of the similarity of the approach, I go fairly quickly through the relevant steps.

PROPOSITION 8.1. *There exists a term $\mathcal{M}$ such that:*

$$\mathbf{L^iD}_{\lambda} \vdash \mathcal{M} = \lambda x_1 x_2 x_3\,((x_2 = 0 \wedge x_3 = 0) \vee$$
$$\bigvee y_1 \bigvee y_2 \bigvee y_2\,(y_1 = x_1 \sqcap y_2' = x_2 \sqcap y_3 + y_1 = x_3 \sqcap \langle\langle y_1, y_2\rangle, y_3\rangle{\in}\mathcal{M}))\,.$$

Proof. As usual, this is an immediate consequence of the fixed-point property. QED

Conventions 8.2. (1) As in the case of addition I introduce an abbreviation to simplify presentation:

$$\mathcal{M}_s :\equiv \lambda x_2 x_3 ((x_2 = 0 \wedge x_3 = 0) \vee \\ \bigvee y_2 \bigvee y_3 (y_2' = x_2 \Box y_3 + y_1 = x_3 \Box \langle\langle s, y_2\rangle, y_3\rangle \in \mathcal{M}))\,.$$

As before, the full definition is only really needed in the proof of one of the substitution properties below.
(2) In order to save space in presentations, I shall occasionally use the following abbreviations:
$bas_{\mathcal{M}_s}[t, r]$ for $t = 0 \wedge r = 0$, and
$stp_{\mathcal{M}_s}[t, r]$ for $\bigvee y_2 \bigvee y_3 (y_2' = t \Box y_2 + s = r \Box \langle\langle s, y_2\rangle, y_3\rangle \in \mathcal{M})$.

Proposition 8.3. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.3i) $bas_{\mathcal{M}_s}[s', r] \Rightarrow$;

(8.3ii) $stp_{\mathcal{M}_s}[0, r] \Rightarrow$.

Proof. As for 6.3i and 6.3ii; left to the reader. QED

Proposition 8.4. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.4i) $s_1 = s_2, t_1 = t_2, r_1 = r_2, \langle\langle s_1, t_1\rangle, r_1\rangle \in \mathcal{M} \Rightarrow \langle\langle s_2, t_2\rangle, r_2\rangle \in \mathcal{M}$;

(8.4ii) $\Rightarrow \langle t, r\rangle \in \mathcal{M}_s \Leftrightarrow \langle\langle s, t\rangle, r\rangle \in \mathcal{M}$.

Proof. As for 6.7. Left to the reader. QED

Proposition 8.5. *Inferences according to the following schema are* $\mathbf{L^iD}_\lambda$*-derivable:*

$$\frac{r_2 = r_1 + s, \langle t, r_1\rangle \in \mathcal{M}_s, \Gamma \Rightarrow C}{stp_{\mathcal{M}_s}[t', r_2], \Gamma \Rightarrow C}\,.$$

Proof. As for 6.5; left to the reader. QED

Definition 8.6. $s \cdot t :\equiv \mathcal{M}_s[\![t]\!]$. I shall use $\mathcal{M}_s[\![t]\!]$ and $(s \cdot t)$ interchangeably.

As for the case of addition, the first thing to establish about this definition is that it is substitutionally transparent.

Proposition 8.7. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.7i) $\quad s = t \Rightarrow s \cdot r = t \cdot r\,;$

(8.7ii) $\quad s = t \Rightarrow r \cdot s = r \cdot t\,.$

Proof. As for 6.7. Left to the reader. QED

Proposition 8.8. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.8i) $\quad \Rightarrow \langle 0, 0\rangle \in \mathcal{M}_s\,;$

(8.8ii) $\quad \langle t, r\rangle \in \mathcal{M}_s \Rightarrow \langle t', r + s\rangle \in \mathcal{M}_s\,.$

Proof. As for 6.8; left to the reader. QED

Proposition 8.9. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.9i) $\quad \Rightarrow s \cdot 0 = 0\,;$

(8.9ii) $\quad \Rightarrow \langle 0, s \cdot 0\rangle \in \mathcal{M}_s\,.$

Proof. As for 6.9; left to the reader. QED

Proposition 8.10. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.10i) $\quad \langle t, s \cdot t\rangle \in \mathcal{M}_s, a \in (s \cdot t') \Rightarrow a \in (s \cdot t + s)\,;$

(8.10ii) $\quad uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, a \in (s \cdot t + s) \Rightarrow a \in (s \cdot t')\,;$

(8.10iii) $\quad uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s \Rightarrow s \cdot t' = s \cdot t + s\,;$

(8.10iv) $\quad uni[t, \mathcal{M}_s], 2\,[\langle t, s \cdot t\rangle \in \mathcal{M}_s] \Rightarrow \langle t', s \cdot t'\rangle \in \mathcal{M}_s\,.$

Proof. Re 8.10i. This follows directly from 8.8ii in the usual way.
Re 8.10ii. For the nonce, let $\mathfrak{M}[s, t, c]$ stand for $\langle t, s \cdot t\rangle \in \mathcal{M}_s \,\Box\, \langle t, c\rangle \in \mathcal{M}_s$:

$$\dfrac{\langle t, s\cdot t\rangle \in \mathcal{M}_s, \langle t, c\rangle \in \mathcal{M}_s \Rightarrow \mathfrak{M}[s,t,c] \qquad \dfrac{a\in(c+s) \Rightarrow a\in(c+s)}{s\cdot t = c, a\in(s\cdot t+s) \Rightarrow a\in(c+s)}}{\mathfrak{M}[s,t,c] \to s\cdot t = c, \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s), \langle t, c\rangle \in \mathcal{M}_s \Rightarrow a\in(c+s)}$$

$$\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s), \langle t, c\rangle \in \mathcal{M}_s \Rightarrow a\in(c+s)}$$

$$\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s), b = c+s, \langle t, c\rangle \in \mathcal{M}_s \Rightarrow a\in b}$$

$$\overline{\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s), \langle t', b\rangle \in \mathcal{M}_s \Rightarrow a\in b}}$$

$$\overline{\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s) \Rightarrow \langle t', b\rangle \in \mathcal{M}_s \to a\in b}}$$

$$\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s) \Rightarrow \bigwedge y(\langle t', y\rangle \in \mathcal{M}_s \to a\in y)}$$

$$\overline{uni[s,t,\mathcal{M}], \langle t, s\cdot t\rangle \in \mathcal{M}_s, a\in(s\cdot t+s) \Rightarrow a\in(s\cdot t')}\ .$$

Re 8.10iii. This is a straightforward consequence of 8.10i and 8.10ii.
Re 8.10iv. As for 6.10iv; left to the reader. QED

PROPOSITION 8.11. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(8.11i) $\Rightarrow uni[0, \mathcal{M}_a]$;

(8.11ii) $uni[t, \mathcal{M}_s] \Rightarrow uni[t', \mathcal{M}_s]$.

Proof. *Re* 8.11i. Almost trivial; left to the reader.
Re 8.11ii. Employ 8.3i and 6.7ii with two inferences according to 8.5:

$$\dfrac{c_1 = c_2 \Rightarrow c_1 + s = c_2 + s}{uni[t, \mathcal{M}_s], \langle t, c_1\rangle \in \mathcal{M}_2, \langle t, c_2\rangle \in \mathcal{M}_2 \Rightarrow c_1 + s = c_2 + s}$$

$$\overline{\overline{uni[t, \mathcal{M}_s], c_1 + s = a, \langle t, c_1\rangle \in \mathcal{M}_2, c_2 + s = b, \langle t, c_2\rangle \in \mathcal{M}_2 \Rightarrow a = b}}$$

$$\dfrac{bas_{\mathcal{M}_s}[t', a] \Rightarrow \qquad \overline{\overline{uni[t, \mathcal{M}_s], stp_{\mathcal{M}_s}[t', a], stp_{\mathcal{M}_s}[t', a] \Rightarrow a = b}}}{uni[t, \mathcal{M}_s], bas_{\mathcal{M}_s}[t', a] \vee stp_{\mathcal{M}_s}[t', a], bas_{\mathcal{M}_s}[t', a] \vee stp_{\mathcal{M}_s}[t', a] \Rightarrow a = b}$$

$$\overline{\overline{uni[t, \mathcal{M}_s], \langle t', a\rangle \in \mathcal{M}_s, \langle t', b\rangle \in \mathcal{M}_s \Rightarrow a = b}}$$

$$\overline{uni[t, \mathcal{M}_s] \Rightarrow uni[t', \mathcal{M}_s]}\ .$$

QED

CONVENTIONS 8.12.
(1) $\mathfrak{C}_{\mathcal{A}_b} :\equiv uni[*_1, \mathcal{A}_b] \,\square\, \langle *_1, b + *_1\rangle \in \mathcal{A}_b \,\square\, (b + *_1) \in \mathbf{N}$.
(2) $\mathfrak{C}_{\mathcal{M}_a} :\equiv a \in \mathbf{N}^\circ \,\square\, uni[*_1, \mathcal{M}_a] \,\square\, \langle *_1, a\cdot *_1\rangle \in \mathcal{M}_a \,\square\, (b\cdot *_1) \in \mathbf{N}$.

Proposition 8.13. *The following is* $\mathbf{L^{i}D}_{\lambda}$*-deducible:*

(8.13i) $uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, \mathfrak{C}_{\mathcal{A}_b}, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N}$;

(8.13ii) $uni[b, \mathcal{M}_a], \langle b, \mathcal{M}_a[\![b]\!] \rangle \in \mathcal{M}_a, (a \cdot b) \in \mathbf{N}, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N}$.

Proof. Re 8.13i. Employ 8.10i:

$$
\begin{array}{c}
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s \Rightarrow s \cdot t' = s \cdot t + s \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, s \cdot t = b \Rightarrow s \cdot t' = b + s \\
\hline
c \in \mathbf{N}^{\circ} \Rightarrow c \in \mathbf{N}^{\circ} \qquad uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, b + s = c, s \cdot t = b \Rightarrow s \cdot t' = c \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, c \in \mathbf{N}^{\circ}, b + s = c, s \cdot t = b \Rightarrow c \in \mathbf{N}^{\circ} \Box\, s \cdot t' = c \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, c \in \mathbf{N}^{\circ}, b + s = c, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, c \in \mathbf{N}^{\circ} \Box\, b + s = c, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, \bigvee y\, (y \in \mathbf{N}^{\circ} \Box\, b + s = y), s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\
\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, (b + s) \in \mathbf{N}, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\
\hline\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, uni[s, \mathcal{A}_b], \langle s, \mathcal{A}_b[\![s]\!] \rangle \in \mathcal{A}_b, (b + s) \in \mathbf{N}, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\
\hline\hline
uni[t, \mathcal{M}_s], \langle t, s \cdot t \rangle \in \mathcal{M}_s, \mathfrak{C}_{\mathcal{A}_b}[s], s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N}
\end{array} .
$$

Re 8.13ii. Employ 8.7i and 8.7ii:

$$
\begin{array}{c}
s = a, t = b \Rightarrow (s \cdot t) = (a \cdot b) \\
\hline
c \in \mathbf{N}^{\circ} \Rightarrow c \in \mathbf{N}^{\circ} \qquad (a \cdot b) = c, s = a, t = b \Rightarrow (s \cdot t) = c \\
\hline
c \in \mathbf{N}^{\circ}, (a \cdot b) = c, s = a, t = b \Rightarrow c \in \mathbf{N}^{\circ} \Box\, (s \cdot t) = c \\
\hline
c \in \mathbf{N}^{\circ}, (a \cdot b) = c, s = a, t = b \Rightarrow \bigvee y (y \in \mathbf{N}^{\circ} \Box\, (s \cdot t) = y) \\
\hline
c \in \mathbf{N}^{\circ}, (a \cdot b) = c, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N} \\
\hline
c \in \mathbf{N}^{\circ} \Box\, (a \cdot b) = c, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N} \\
\hline
\bigvee y (y \in \mathbf{N}^{\circ} \Box\, (a \cdot b) = y), s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N} \\
\hline
(a \cdot b) \in \mathbf{N}, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N} \\
\hline\hline
a \in \mathbf{N}^{\circ}, uni[b, \mathcal{M}_a], \langle b, \mathcal{M}_a[\![b]\!] \rangle \in \mathcal{M}_a, (a \cdot b) \in \mathbf{N}, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N} \\
\hline\hline
\mathfrak{C}_{\mathcal{M}_a}, s = a, t = b \Rightarrow (s \cdot t) \in \mathbf{N}
\end{array} .
$$

QED

Proposition 8.14. *The following is* $\mathbf{L^i D}_\lambda$*-deducible:*

(8.14i) $\Rightarrow (s \cdot 0) \in \mathbf{N}\,;$

(8.14ii) $a \in \mathbf{N}^\circ \Rightarrow a \in \mathbf{N}^\circ \Box\, uni[0, \mathcal{M}_a] \Box \langle 0, a \cdot 0\rangle \in \mathcal{M}_a \Box (a \cdot 0) \in \mathbf{N}\,.$

Proof. Re 8.14ii. Employ 4.6i and 8.9i:

$$\begin{array}{c} \Rightarrow 0 \in \mathbf{N}^\circ \qquad \Rightarrow a \cdot 0 = 0 \\ \hline \Rightarrow 0 \in \mathbf{N}^\circ \Box (a \cdot 0) = 0 \\ \hline \Rightarrow \bigvee y (y \in \mathbf{N}^\circ \Box (a \cdot 0) = y) \\ \hline \Rightarrow (a \cdot 0) \in \mathbf{N} \end{array} \,.$$

Re 8.14ii. This is a simple conjunction of 8.11i, 8.9ii and 8.14i. QED

Proposition 8.15. *The following is* $\mathbf{L^i D}_\lambda^{\mathsf{Z}\restriction 9}$*-deducible:*

(8.15i) $s \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, (s \cdot t) \in \mathbf{N} \Rightarrow (s \cdot t') \in \mathbf{N}\,;$

(8.15ii) $3[s \in \mathbf{N}^\circ \Box\, uni[t, \mathcal{M}_s] \Box \langle t, s \cdot t\rangle \in \mathcal{M}_s \Box (s \cdot t) \in \mathbf{N}] \Rightarrow \mathfrak{C}_{\mathcal{M}_s}[t']\,.$

Proof. Re 8.15i. In the first line let $\mathcal{F}_1 :\equiv uni[t, \mathcal{M}_s]$ and $\mathcal{F}_2 :\equiv \langle t, s \cdot t\rangle \in \mathcal{M}_s$. Employ 7.3i, 7.3ii, and 8.13i:

$$\begin{array}{c} b \in \mathbf{N}^\circ \Rightarrow \mathfrak{C}_{\mathcal{A}_b}[0] \quad 3[\mathfrak{C}_{\mathcal{A}_b}[c]] \Rightarrow \mathfrak{C}_{\mathcal{A}_b}[c'] \quad \mathfrak{C}_{\mathcal{A}_b}[s], \mathcal{F}_1, \mathcal{F}_2, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\ \hline s \in \mathbf{N}^\circ, b \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\ \hline s \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, b \in \mathbf{N}^\circ, s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\ \hline s \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, b \in \mathbf{N}^\circ \Box s \cdot t = b \Rightarrow (s \cdot t') \in \mathbf{N} \\ \hline s \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, \bigvee y\,(y \in \mathbf{N}^\circ \Box s \cdot t = y) \Rightarrow (s \cdot t') \in \mathbf{N} \\ \hline s \in \mathbf{N}^\circ, uni[t, \mathcal{M}_s], \langle t, s \cdot t\rangle \in \mathcal{M}_s, (s \cdot t) \in \mathbf{N} \Rightarrow (s \cdot t') \in \mathbf{N} \end{array} \;{\scriptstyle +9}\,.$$

Re 8.15ii. Essentially a conjunction of 8.11ii, 8.10iv, and 8.15i; left to the reader. QED

Proposition 8.16. $\mathbf{L^i D}_\lambda^{\mathsf{Z}\restriction 18} \vdash\ s \in \mathbf{N}, t \in \mathbf{N} \Rightarrow (s \cdot t) \in \mathbf{N}\,.$

Proof. Employ 8.14ii, 8.15ii, and 8.13ii with an inference according to 4.7viii:

$$\dfrac{a\in\mathbf{N}^\circ \Rightarrow \mathfrak{C}_{\mathcal{M}_a}[0] \qquad \begin{matrix}\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction 9}\\ \vdots \\ 3[\mathfrak{C}_{\mathcal{M}_a}[c]] \Rightarrow \mathfrak{C}_{\mathcal{M}_a}[c']\end{matrix} \qquad \mathfrak{C}_{\mathcal{M}_a}[b], s=a, t=b \Rightarrow (s\cdot t)\in\mathbf{N}}{\dfrac{\dfrac{b\in\mathbf{N}^\circ, a\in\mathbf{N}^\circ, s=a, t=b\Rightarrow (s\cdot t)\in\mathbf{N}}{a\in\mathbf{N}^\circ \square\, s=a, b\in\mathbf{N}^\circ \square\, t=b \Rightarrow (s\cdot t)\in\mathbf{N}}}{\bigvee y(y\in\mathbf{N}^\circ \square\, s=y), \bigvee y(y\in\mathbf{N}^\circ \square\, t=y) \Rightarrow (s\cdot t)\in\mathbf{N}}}\;{+9}\ .$$

QED

Remark 8.17. While in the case of addition, $s\in\mathbf{N}^\circ \Rightarrow s'\in\mathbf{N}^\circ$ was sufficient for proving the totality (cf. 7.2ii above), a proof of the totality of multiplication also requires $s\in\mathbf{N}, t\in\mathbf{N} \Rightarrow (s+t)\in\mathbf{N}$, *i.e.*, the totality of addition. This is what makes the number of **Z**-inferences go up.

9. Exponentiation

Given the treatment of addition and multiplication, I can dispose fairly quickly of exponentiation. Many of the following propositions will only be listed without proof.

Proposition 9.1. *There exists a term $\mathcal{E}$ satisfying:*

$$\begin{aligned}\mathcal{E} = \lambda x_1x_2x_3\,(&(x_2=0\wedge x_3=1)\vee \\ &\bigvee y_1\bigvee y_2\bigvee y_3\,(x_1=y_1\,\square\, x_2=y_2'\,\square\, x_3=y_3\cdot y_1\,\square\,\langle\langle y_1,y_2\rangle, y_3\rangle\in\mathcal{E})\,.\end{aligned}$$

Proof. As usual, this is an immediate consequence of the fixed-point property. QED

Conventions 9.2. (1) The following abbreviation is introduced to simplify presentation:

$$\begin{aligned}\mathcal{E}_s :\equiv \lambda x_2x_3\,(&(x_2=0\wedge x_3=1)\vee \\ &\bigvee y_1\bigvee y_2\bigvee y_3\,(y_1=s\,\square\, y_2'=x_2\,\square\, y_3\cdot y_1=x_3\,\square\,\langle\langle y_1,y_2\rangle, y_3\rangle\in\mathcal{E})\,.\end{aligned}$$

(2) In order to save space in presentations, I shall occasionally use the following abbreviations:

$bas_{\mathcal{E}_s}[t, r]$ for $t = 0 \wedge r = 1$, and
$stp_{\mathcal{E}_s}[t, r]$ for

$$\bigvee y_1 \bigvee y_2 \bigvee y_3 (y_1 = s \,\square\, y_2' = x_2 \,\square\, y_3 \cdot y_1 = x_3 \,\square\, \langle\langle y_1, y_2\rangle, y_3\rangle \in \mathcal{E}) \,.$$

Proposition 9.3. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.3i) $bas_{\mathcal{E}_s}[s', r] \Rightarrow$;

(9.3ii) $stp_{\mathcal{E}_s}[0, r] \Rightarrow$.

Definition 9.4. $s^t :\equiv \mathcal{E}_s[\![t]\!]$. I shall use $\mathcal{E}_s[\![t]\!]$ and s^t interchangeably.

Proposition 9.5. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.5i) $s = t \Rightarrow s^r = t^r$;

(9.5ii) $s = t \Rightarrow r^s = r^t$.

Proposition 9.6. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.6i) $\Rightarrow \langle 0, 1\rangle \in \mathcal{E}_s$;

(9.6ii) $\langle t, r\rangle \in \mathcal{E}_s \Rightarrow \langle t', r \cdot s\rangle \in \mathcal{E}_s$.

Proposition 9.7. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.7i) $\Rightarrow s^0 = 1$;

(9.7ii) $\Rightarrow \langle 0, s^0\rangle \in \mathcal{E}_s$.

Proposition 9.8. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.8i) $uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^{t'}) \Rightarrow a \in (s^t \cdot s)$;

(9.8ii) $uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s) \Rightarrow a \in (s^{t'})$;

(9.8iii) $uni[t, \mathcal{E}_s], \langle t, s^t\rangle \in \mathcal{E}_s \Rightarrow s^{t'} = s^t \cdot s$;

(9.8iv) $uni[t, \mathcal{E}_s], 2[\langle t, s^t\rangle \in \mathcal{E}_s] \Rightarrow \langle t', s^{t'}\rangle \in \mathcal{E}_s$.

Proof. As usual. I only treat 9.8ii.

$$\begin{array}{c}
\langle t, s^t\rangle \in \mathcal{E}_s, \langle t, b\rangle \in \mathcal{E}_s \Rightarrow \langle t, s^t\rangle \in \mathcal{E}_s \,\square\, \langle t, c\rangle \in \mathcal{E}_s \qquad \dfrac{a \in (c \cdot s) \Rightarrow a \in (c \cdot s)}{s^t = c, a \in (s^t \cdot s) \Rightarrow a \in (c \cdot s)} \\
\hline
\langle t, s^t\rangle \in \mathcal{E}_s \,\square\, \langle t, c\rangle \in \mathcal{E}_s \to s^t = c, \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s), \langle t, c\rangle \in \mathcal{E}_s \Rightarrow a \in (c \cdot s) \\
\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s), \langle t, c\rangle \in \mathcal{E}_s \Rightarrow a \in (c \cdot s) \\
\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s), b = c \cdot s, \langle t, c\rangle \in \mathcal{E}_s \Rightarrow a \in b \\
\hline\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s), \langle t', b\rangle \in \mathcal{E}_s \Rightarrow a \in b \\
\hline\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s) \Rightarrow \langle t', b\rangle \in \mathcal{E}_s \to a \in b \\
\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s) \Rightarrow \bigwedge y\, (\langle t', y\rangle \in \mathcal{E}_s \to a \in y) \\
\hline
uni[s, t, \mathcal{E}], \langle t, s^t\rangle \in \mathcal{E}_s, a \in (s^t \cdot s) \Rightarrow a \in (s^{t'})
\end{array} .$$

QED

PROPOSITION 9.9. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.9i) $\Rightarrow uni[0, \mathcal{E}_s]$;

(9.9ii) $uni[t, \mathcal{E}_s] \Rightarrow uni[t', \mathcal{E}_s]$.

CONVENTION 9.10. $\mathfrak{C}_{\mathcal{E}_a} :\equiv a \in \mathbf{N}^\circ \,\square\, uni[*_1, \mathcal{E}_a] \,\square\, \langle *_1, a^{*_1}\rangle \in \mathcal{E}_a \,\square\, a^{*_1} \in \mathbf{N}$.

PROPOSITION 9.11. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.11i) $uni[t, \mathcal{E}_s], \langle t, s^t\rangle \in \mathcal{E}_s, \mathfrak{C}_{\mathcal{M}_b}[s], s^t = b \Rightarrow s^{t'} \in \mathbf{N}$;

(9.11ii) $a \in \mathbf{N}^\circ \,\square\, uni[b, \mathcal{E}_a] \,\square\, \langle x, a^b\rangle \in \mathcal{E}_a \,\square\, a^b \in \mathbf{N}, s = a, t = b \Rightarrow s^t \in \mathbf{N}$.

PROPOSITION 9.12. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(9.12i) $\Rightarrow a^0 \in \mathbf{N}$;

(9.12ii) $a \in \mathbf{N}^\circ \Rightarrow a \in \mathbf{N}^\circ \,\square\, uni[0, \mathcal{E}_a] \,\square\, \langle 0, a^0\rangle \in \mathcal{E}_a \,\square\, a^0 \in \mathbf{N}$.

Proof. *Re* 9.12i. Employ 9.7i:

$$\begin{array}{c}
\Rightarrow 1 \in \mathbf{N}^\circ \qquad \Rightarrow a^0 = 1 \\
\hline
\Rightarrow 1 \in \mathbf{N}^\circ \,\square\, a^0 = 1 \\
\hline
\Rightarrow \bigvee y\, (y \in \mathbf{N}^\circ \,\square\, a^0 = y) \\
\hline
\Rightarrow a^0 \in \mathbf{N}
\end{array} .$$

QED

Proposition 9.13. *The following is* $\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction 18}$*-deducible:*

(9.13i) $s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, s^t\in\mathbf{N} \Rightarrow s^{t'}\in\mathbf{N}$;

(9.13ii) $3[s\in\mathbf{N}^\circ \Box\, uni[t,\mathcal{E}_s] \Box \langle t, s^t\rangle\in\mathcal{E}_s \Box\, s^t\in\mathbf{N}]$

$$\Rightarrow s\in\mathbf{N}^\circ \Box\, uni[t',\mathcal{E}_s] \Box \langle t', s^{t'}\rangle\in\mathcal{E}_s \Box\, s^{t'}\in\mathbf{N}\,.$$

Proof. Re 9.13i. Employ 8.14ii, 8.15ii, and 9.11i with an inference according to 4.7ii. In the first line, let $\mathfrak{E}[s,t]$ stand for $uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s$:

$$\dfrac{\dfrac{\dfrac{\dfrac{b\in\mathbf{N}^\circ \Rightarrow \mathfrak{C}_{\mathcal{M}_b}[0] \qquad \begin{matrix}\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction 9}\\ \vdots \\ 3[\mathfrak{C}_{\mathcal{M}_b}[c]] \Rightarrow \mathfrak{C}_{\mathcal{M}_b}[c']\end{matrix} \qquad \mathfrak{E}[s,t], \mathfrak{C}_{\mathcal{M}_b}[s], s^t=b \Rightarrow s^{t'}\in\mathbf{N}}{s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, b\in\mathbf{N}^\circ, s^t=b \Rightarrow s^{t'}\in\mathbf{N}}\,{\scriptstyle +9}}{s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, b\in\mathbf{N}^\circ \Box\, s^t=b \Rightarrow s^{t'}\in\mathbf{N}}}{s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, \bigvee y\,(y\in\mathbf{N}^\circ \Box\, s^t=y) \Rightarrow s^{t'}\in\mathbf{N}}}{s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, s^t\in\mathbf{N} \Rightarrow s^{t'}\in\mathbf{N}}\,.$$

Re 9.13ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction 18}\\ \vdots \\ s\in\mathbf{N}^\circ, uni[t,\mathcal{E}_s], \langle t, s^t\rangle\in\mathcal{E}_s, s^t\in\mathbf{N} \Rightarrow s^{t'}\in\mathbf{N}\end{matrix}}{s\in\mathbf{N}^\circ, 2[uni[t,\mathcal{E}_s]], 2[\langle t, s^t\rangle\in\mathcal{E}_s], s^t\in\mathbf{N} \Rightarrow \langle t', s^{t'}\rangle\in\mathcal{E}_s \Box\, s^{t'}\in\mathbf{N}}}{s\in\mathbf{N}^\circ, 3[uni[t,\mathcal{E}_s]], 2[\langle t, s^t\rangle\in\mathcal{E}_s], s^t\in\mathbf{N} \Rightarrow uni[t,\mathcal{E}_s] \Box \langle t', s^{t'}\rangle\in\mathcal{E}_s \Box\, s^{t'}\in\mathbf{N}}}{s\in\mathbf{N}^\circ, 3[uni[t,\mathcal{E}_s]], 3[\langle t, s^t\rangle\in\mathcal{E}_s], 3[s^t\in\mathbf{N}] \Rightarrow uni[t,\mathcal{E}_s] \Box \langle t', s^{t'}\rangle\in\mathcal{E}_s \Box\, s^{t'}\in\mathbf{N}}}{3[s\in\mathbf{N}^\circ \Box\, uni[t,\mathcal{E}_s] \Box \langle t, s^t\rangle\in\mathcal{E}_s \Box\, s^t\in\mathbf{N}] \Rightarrow s\in\mathbf{N}^\circ \Box\, uni[t,\mathcal{E}_s] \Box \langle t', s^{t'}\rangle\in\mathcal{E}_s \Box\, s^{t'}\in\mathbf{N}}\,.$$

QED

Proposition 9.14. $\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction 27} \vdash\ s\in\mathbf{N}, t\in\mathbf{N} \Rightarrow s^t\in\mathbf{N}\,.$

Proof. Employ 9.12ii, 9.13ii, and 9.11ii with an inference according to 4.7viii:

$$
\dfrac{a\in\mathbf{N}^{\circ}\Rightarrow\mathfrak{C}_{\mathcal{E}_a}[0] \qquad \begin{array}{c}\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}^{\mathrm{Z}\restriction 18}\\ \vdots \\ 3[\mathfrak{C}_{\mathcal{E}_a}[c]]\Rightarrow\mathfrak{C}_{\mathcal{E}_a}[c']\end{array} \qquad \mathfrak{C}_{\mathcal{E}_a}[b], s=a, t=b\Rightarrow(s^t)\in\mathbf{N}}{\dfrac{\dfrac{\dfrac{a\in\mathbf{N}^{\circ}, s=a, b\in\mathbf{N}^{\circ}, t=b\Rightarrow s^t\in\mathbf{N}}{a\in\mathbf{N}^{\circ}\Box\, s=a, b\in\mathbf{N}^{\circ}\Box\, t=b\Rightarrow s^t\in\mathbf{N}}}{\bigvee y(y\in\mathbf{N}^{\circ}\Box\, s=y), \bigvee y(y\in\mathbf{N}^{\circ}\Box\, t=y)\Rightarrow s^t\in\mathbf{N}}}{s\in\mathbf{N}, t\in\mathbf{N}\Rightarrow s^t\in\mathbf{N}}}\;{\scriptstyle +9} \; .
$$

QED

Remark 9.15. The totality of exponentiation presupposes that of addition and multiplication: that adds up to 27 **Z**-inferences.

10. The series of primitive recursive functions continued

So far I have considered the following two-place primitive recursive functions:

$$\phi_0(a,b)=a+b\,,$$
$$\phi_1(a,b)=a\cdot b\,,$$
$$\phi_2(a,b)=a^b\,.$$

They can be seen as forming the beginning part of a series:[24] just as multiplication is the iteration of addition in the form

$$a+a+\ldots+a=a\cdot\mathrm{n}\,,$$

exponentiation is the iteration of multiplication in the form

$$a\cdot a\cdot\ldots\cdot a=a^{\mathrm{n}}\,.$$

This can be continued to a super-exponentiation:

$$a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$$

[24] As has been done in [12], p. 185 ([27], p. 388), and also [13], p. 336, to motivate the formulation of the Ackermann function.

which would be governed by the following recursion equations:

$$\phi_3(a,0) = a\,,$$
$$\phi_3(a,b') = a^{\phi_3(a,b)}\,, \quad \text{or:} \quad \phi_2(a,\phi_3(a,b))\,.$$

In general, a series of functions ϕ_{n} can be defined for $\mathrm{n} > 2$ as follows:

$$\phi_{\mathrm{n}'}(a,0) = a\,,$$
$$\phi_{\mathrm{n}'}(a,b') = \phi_{\mathrm{n}}(a,\phi_{\mathrm{n}'}(a,b))\,.$$

In terms of their recursion equations, we have the following:

$$\phi_0(a,b) = \begin{cases} a\,, & \text{if } b = 0\,; \\ \phi_0(a,c)'\,, & \text{if } b = c' \text{ for some } c\,. \end{cases}$$

$$\phi_1(a,b) = \begin{cases} 0\,, & \text{if } b = 0\,; \\ \phi_0(\phi_1(a,c),a)\,, & \text{if } b = c' \text{ for some } c\,. \end{cases}$$

$$\phi_2(a,b) = \begin{cases} 1\,, & \text{if } b = 0\,; \\ \phi_1(\phi_2(a,c),a)\,, & \text{if } b = c' \text{ for some } c\,. \end{cases}$$

$$\phi_3(a,b) = \begin{cases} a\,, & \text{if } b = 0\,; \\ \phi_2(\phi_3(a,c),a)\,, & \text{if } b = c' \text{ for some } c\,. \end{cases}$$

$$\vdots$$

$$\phi_{\mathrm{n}'}(a,b) = \begin{cases} a\,, & \text{if } b = 0\,; \\ \phi_{\mathrm{n}}(\phi_{\mathrm{n}'}(a,c),a)\,, & \text{if } b = c' \text{ for some } c\,. \end{cases}$$

The recursion equations for the functions ϕ_{n} are $\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction_4}$-deducible, and that for all $\mathrm{n} \in \mathbb{N}$. It is the proof of the totality of ϕ_{n} that requires recourse to the totality of the functions ϕ_{k} with $\mathrm{k} < \mathrm{n}$ and thus can be expected to be $\mathbf{L^iD}_\lambda^{\mathrm{Z}\restriction_{9\mathrm{n}+9}}$-deducible.

Now it is well-known that Ackermann's function can be presented as a kind of totalization over this series, by turning the index number of the ϕ_{k} into an additional argument. Its recursion equations are no longer $\mathbf{L^iD}_\lambda^{\mathrm{Z}}$-deducible; we need something like the reinforced necessity operator that I introduced in [23], pp. 136–159. But this will be the topic of a follow-up to the present paper in which I will consider the complexity of k-recursive functions in more detail.

11. Z-inferences as a measure of complexity[25]

Towards the end of his famous address entitled *Über das Unendliche* ("On the infinite"), Hilbert declared (in translation):[26]

> The role that remains to the infinite is ⟦...⟧ merely that of an idea—if, in accordance with Kant's words, we understand by an idea a concept of reason that transcends all experience and through which the concrete is completed so as to form a totality⟦.⟧[27]

Hilbert's proof theory was meant to justify the use of classical logic for this supposed role of infinity as "merely that of an idea", but, cautiously put, his program was not successful. If this is taken to indicate that the role of "a concept of reason that transcends all experience" cannot simply be reduced to that of a neutral supplementation, then the question regarding the nature of the infinite and its appropriate logic would have to be raised again.

Intuitionistic logic, despite its declared aim to overcome classical logic in its treatment of the infinite is not a suitable alternative: it remains within a somewhat classical paradigm. As Girard put it:

> Classical and intuitionistic logics deal with *stable* truths:
>
> If A and A $\Rightarrow$ B , then B, *but A still holds.*[28]

This is a hallmark of contraction and that's why abandoning contraction recommends itself when confronting the possibility of unstable truths — something that may well happen when dealing with infinity.

With contraction available, resources can be multiplied *ad libitum* at no extra costs.[29] This proves vital when it comes to formulating a term that is to capture *exactly* the natural numbers.[30] The induction

[25] The idea put forward in this section is very tentative, indeed, and should be taken with a pound of salt. In any case, it is what motivated my investigations into how many **Z**-inferences are needed to prove the totality of certain primitive recursive functions.

[26] [27], p. 367.

[27] [27], p. 392.

[28] [9], p. 1.

[29] In Girard's diction "*contraction* is the fingernail of infinity" ([7], p. 78).

[30] It is easy enough to provide a term that captures all natural numbers, but the point for induction is that it is *only* the natural numbers that are captured. This is what I labeled "exclusion principle" in remarks 116.6 and 119.1 in [21], for example.

step is available as often as one likes, but it only has to be accounted for once. Without contraction this changes: with assumptions having to be accounted for, the formulation of the induction step requires special attention to the effect of specifying how often it is available. This is what **Z**-inferences were designed to accomplish.[31] They provide an alternative approach to infinity — and this approach is what I want to propose as a basis for a measure of complexity: how many **Z**-inferences are required in the proof of a particular result.

In view of these considerations I should emphasize that my approach is not so much aimed at a notion of *computational* complexity, but more at something like a *metaphysical* complexity.

Now recall the results of the foregoing sections:[32]

1. The totality of the predecessor function is $\mathbf{L^iD}_\lambda^{Z\upharpoonright 1}$-deducible;
2. the recursion equations of primitive recursive functions are $\mathbf{L^iD}_\lambda^{Z\upharpoonright 4}$-deducible;
3. the totality of addition is $\mathbf{L^iD}_\lambda^{Z\upharpoonright 9}$-deducible;
4. the totality of multiplication is $\mathbf{L^iD}_\lambda^{Z\upharpoonright 18}$-deducible;
5. the totality of exponentiation is $\mathbf{L^iD}_\lambda^{Z\upharpoonright 27}$-deducible;
6. In general, the totality of φ_n can be expected to be $\mathbf{L^iD}_\lambda^{Z\upharpoonright 9n+9}$-deducible.

How can this be linked to a notion of complexity? Not surprisingly, perhaps, the suggestion I want to make evokes consistency proofs. The idea is that the number of **Z**-inferences in deductions determines how high an induction is needed for a consistency proof. As is well-known, for the system which allows no **Z**-inference at all ($\mathbf{L^iD}_\lambda$), an induction up to ω suffices, but for the realm beyond that I need a conjecture.

CONJECTURE 11.1. *The consistency of* $\mathbf{L^iD}_\lambda^{Z\upharpoonright n}$ *can be established by an induction up to* ω^{n+1}.

Comment. This would be in accordance with the consistency of $\mathbf{L^iD}_\lambda^{Z}$ being ω^ω-provable: every $\mathbf{L^iD}_\lambda^{Z}$-deduction is a $\mathbf{L^iD}_\lambda^{Z\upharpoonright n}$ for some $n \in \mathbb{N}$.

[31] There are alternative ways of doing this such as [16] and [19] neither of which, however, is designed to capture full primitive recursion.

[32] It must be understood that the following deducibility claims indicate upper bounds only, *i.e.*, I haven't established that any of the proofs, be that of the recursion equations or the totality of addition, multiplication, or exponentiation cannot be reduced to less **Z**-inferences.

In view of the foregoing conjecture, the following hierarchy is suggested by taking the proof of the totality of a function as the basis for a measure of complexity:

1. the predecessor function is assigned the complexity ω^2;
2. addition is of a complexity $\geq \omega^2$ and $\leq \omega^{10}$;
3. multiplication is of a complexity $\geq \omega^3$ and $\leq \omega^{19}$;
4. exponentiation is of a complexity $\geq \omega^4$ and $\leq \omega^{28}$;
5. in general, the function φ_{n} is of a complexity $\geq \omega^{\mathrm{n}+1}$ and $\leq \omega^{9\mathrm{n}+10}$.

If this is continued as suggested at the end of the last section, the complexity of a function defined by nested double recursion can be expected to be somewhere above ω^ω. Thus the measure of complexity suggested here differs quite significantly from the one suggested by Rózsa Péter's work according to which nested n-fold recursion would have a complexity of ω^n. This difference has its origin in a different treatment of infinity as expressed in the formulation of the term $\mathbf{N}^\circ$.

What remains is the question of whether this hierarchy is immune to the possibilities of reducing an induction up to ω^2, for instance, to an ordinary one. Of course, my immediate response would be to direct attention, once again, to the different treatment of infinity. The point is simply that these reductions are based on a *classical* form of induction, *i.e.*, one involving classical logic, in particular contraction, albeit on a meta level. On the basis of the present resource conscious logic, not even course-of-value induction (or: strong induction) is reducible to ordinary induction. In the classical case (of suitable higher order), course-of-value induction can be established in the form

$$s \in \mathbf{N}, \bigwedge y (\bigwedge x (x < y \to \mathfrak{F}[x]) \to \mathfrak{F}[y] \Rightarrow \mathfrak{F}[s] \,,$$

where $\mathbf{N}$ is the term

$$\lambda x \bigwedge y (\bigwedge z (z \in y \to z' \in y) \to (0 \in y \to x \in y)) \,,$$

whereas its dialectical counterpart requires a necessity operator:[33]

$$s \in \mathbf{N}^\circ, \Box \bigwedge y (\bigwedge x (x < y \to \mathfrak{F}[x]) \to \mathfrak{F}[y] \Rightarrow \mathfrak{F}[s] \,.$$

[33] Cf. [22], p. 676. 7iii.

Differently put, if strong induction (in its classical form) is captured by the term

$$\mathbf{N}^{+} :\equiv \lambda x \bigwedge y (\bigwedge z (\bigwedge z_1 (z_1 < z \rightarrow z_1 \in y) \rightarrow z \in y) \rightarrow x \in y)\,,$$

then the following is classically provable:

$$(11.48) \qquad s \in \lambda x \bigwedge y (\bigwedge z (z \in y \rightarrow z' \in y) \rightarrow (0 \in y \rightarrow x \in y)) \Rightarrow s \in \mathbf{N}^{+}\,.$$

To see this, take $\mathfrak{C} :\equiv \bigwedge y (y < *_1 \rightarrow y \in \mathbf{N}^{+})$ and confirm that the following is provable, classically as well as dialectically:

$$\Rightarrow \mathfrak{C}[0]\,,$$
$$\mathfrak{C}[a], \bigwedge x (\mathfrak{C}[x] \rightarrow x \in \mathbf{N}^{+}) \Rightarrow \mathfrak{C}[a']\,,$$
$$\mathfrak{C}[s'] \Rightarrow s \in \mathbf{N}^{+}\,.$$

In the classical case, this yields 11.48 by means of a simple induction, but not so in the dialectical case. It's the side wff $\bigwedge x (\mathfrak{C}[x] \rightarrow x \in \mathbf{N}^{+})$ which makes things more complicated. Instead of a proof of 11.48 by means of a simple induction one only gets

$$\mathbf{L^{i}D}_{\lambda}^{\mathsf{Z}\restriction 2} \vdash \Rightarrow \mathbf{N}^{\circ} \subseteq \lambda x \bigwedge y (\Box \bigwedge z (\bigwedge z_1 (z_1 < z \rightarrow z_1 \in y) \rightarrow z \in y) \rightarrow x \in y)\,,$$

i.e., the reduction becomes more costly of deductive means. This is resource consciousness manifesting itself.

This, I contend, matters in the case of induction up to ω^2 (and beyond, of course) as well. To be sure, this is not meant to serve as a *proof* of the impossibility of reducing induction up to ω^2 to ordinary induction in contraction free logic, but just to indicate, how a familiar classical strategy may turn sour in the case of contraction free logic: a reduction of induction up to ω^2 to an induction up to ω may require an induction up to ω^2 — provided, of course, one works within a contraction free logic.

12. Appendix: Natural numbers and elements of $\mathbf{\Psi}$

The term $\mathbf{N}^{\circ}$, which is designed to represent the set of natural numbers on the formal level, has been introduced *via* a notion of weak implication which, in turn, was based on a notion of having available a certain wff a certain *number of times*. Having available a wff a certain *number* of times, however, does not require a full-fledged notion of natural number, but only that of a certain *proto-number*, elements of the collection $\mathbf{\Psi}$, which was captured in the formal notion $\breve{\mathbf{\Pi}}^{\circ}$.

In definition 2.10 (2) above, a correspondence has been introduced which provided a link between natural numbers (*i.e.*, elements of $\mathbb{N}$) and elements of $\boldsymbol{\Psi}$. I shall now provide a term that will take care of this correspondence on the formal level, *i.e.*, relate $\breve{\mathbf{\Pi}}$ and $\mathbf{N}^{\circ}$. In character this term resembles a primitive recursive function, only that it doesn't have values in the natural numbers, but Ψ instead.

PROPOSITION 12.1. *There exists a fixed point* $\tilde{\mathrm{v}}$ *such that*

$$\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda} \vdash \; \tilde{\mathrm{v}} = \lambda x_1 x_2 ((x_1 = 0' \mathbin{\square} x_2 = I) \vee \\ \bigvee y_1 \bigvee y_2 (x_1 = y_1'' \mathbin{\square} x_2 = y_2^I \mathbin{\square} \langle y_1', y_2 \rangle \in \tilde{\mathrm{v}}))\,.$$

The treatment is essentially the same as for addition (treated as a one-place function) only that the values are not in the natural numbers. I just list the relevant properties without proof.

PROPOSITION 12.2. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(12.2i) $\Rightarrow \langle 0', I \rangle \in \tilde{\mathrm{v}}\,;$

(12.2ii) $\langle s', t \rangle \in \tilde{\mathrm{v}} \Rightarrow \langle s'', t^I \rangle \in \tilde{\mathrm{v}}\,.$

PROPOSITION 12.3. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(12.3i) $\Rightarrow \tilde{\mathrm{v}}[\![0']\!] = I\,;$

(12.3ii) $\Rightarrow \langle 0', \tilde{\mathrm{v}}[\![0']\!] \rangle \in \tilde{\mathrm{v}}\,.$

PROPOSITION 12.4. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(12.4i) $uni[s', \tilde{\mathrm{v}}], \langle s', \tilde{\mathrm{v}}[\![s']\!] \rangle \in \tilde{\mathrm{v}} \Rightarrow \tilde{\mathrm{v}}[\![s'']\!] = \tilde{\mathrm{v}}[\![s']\!]^I\,;$

(12.4ii) $uni[s', \tilde{\mathrm{v}}], \langle s', \langle s', \tilde{\mathrm{v}}[\![s']\!] \rangle \in \tilde{\mathrm{v}} \Rightarrow \langle s'', \tilde{\mathrm{v}}[\![s'']\!] \rangle \in \tilde{\mathrm{v}}\,.$

PROPOSITION 12.5. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(12.5i) $\Rightarrow uni[0', \tilde{\mathrm{v}}]\,;$

(12.5ii) $uni[s', \tilde{\mathrm{v}}] \Rightarrow uni[s'', \tilde{\mathrm{v}}]\,.$

PROPOSITION 12.6. *The following is* $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$*-deducible:*

(12.6i) $\Rightarrow uni[0', \tilde{\mathrm{v}}] \mathbin{\square} \langle 0', I \rangle \in \tilde{\mathrm{v}}\,;$

(12.6ii) $2\,[uni[a, \tilde{\mathrm{v}}] \mathbin{\square} \langle a, \tilde{\mathrm{v}}[\![a]\!] \rangle \in \tilde{\mathrm{v}}] \Rightarrow uni[a', \tilde{\mathrm{v}}] \mathbin{\square} \langle a', \tilde{\mathrm{v}}[\![a']\!] \rangle \in \tilde{\mathrm{v}}\,.$

PROPOSITION 12.7. $\mathbf{L^iD}_\lambda^{\mathsf{Z}\restriction 4} \vdash \; s \in \mathbf{N}^\circ \Rightarrow \tilde{\mathrm{v}}\llbracket s'' \rrbracket = \tilde{\mathrm{v}}\llbracket s' \rrbracket^I$.

Proof. As usual, employ an inference according to schema 4.7vii, in the present case with 12.6i and 12.6ii. QED

Next comes totality in the sense of showing that $s' \in \mathbf{N}^\circ \Rightarrow \tilde{\mathrm{v}}\llbracket s' \rrbracket \in \breve{\boldsymbol{\Pi}}$.

PROPOSITION 12.8. *The following is* $\mathbf{L^iD}_\lambda$*-deducible:*

(12.8i) $\Rightarrow uni[0', \tilde{\mathrm{v}}] \,\square\, \langle 0', \tilde{\mathrm{v}}\llbracket x \rrbracket \rangle \in \tilde{\mathrm{v}} \,\square\, \tilde{\mathrm{v}}\llbracket 0' \rrbracket \in \breve{\boldsymbol{\Pi}}$;

(12.8ii) $3\,[uni[a', \tilde{\mathrm{v}}] \,\square\, \langle a', \tilde{\mathrm{v}}\llbracket a' \rrbracket \rangle \in \tilde{\mathrm{v}} \,\square\, \tilde{\mathrm{v}}\llbracket a' \rrbracket \in \breve{\boldsymbol{\Pi}}] \Rightarrow$

$$uni[a'', \tilde{\mathrm{v}}] \,\square\, \langle a'', \tilde{\mathrm{v}}\llbracket a'' \rrbracket \rangle \in \tilde{\mathrm{v}} \,\square\, \tilde{\mathrm{v}}\llbracket a'' \rrbracket \in \breve{\boldsymbol{\Pi}} .$$

PROPOSITION 12.9. $\mathbf{L^iD}_\lambda^{\mathsf{Z}\restriction 9} \vdash \; s' \in \mathbf{N}^\circ \Rightarrow \tilde{\mathrm{v}}\llbracket s' \rrbracket \in \breve{\boldsymbol{\Pi}}$.

Proof. As usual, employ an inference according to schema 4.7viii, in the present case with 12.8i and 12.8ii. QED

PROPOSITION 12.10. $\mathbf{L^iD}_\lambda \vdash \Rightarrow \tilde{\mathrm{v}}\llbracket 0 \rrbracket = \mathcal{V}$

Proof. Straightforward; left to the reader. QED

REMARKS 12.11. (1) It will be obvious that $[A/\tilde{\mathrm{v}}\llbracket n' \rrbracket] \leftrightarrow [A]^{\cdot \mathrm{n}}$ can be established by a meta-theoretical induction on n. In other words, the meta-theoretical notion of $[A]^{\cdot \mathrm{n}}$ can be replaced by the formal notion $[A/\tilde{\mathrm{v}}\llbracket n' \rrbracket]$.
(2) What 12.10 says is basically that the function $\tilde{\mathrm{v}}$ is not "defined" for the argument 0 in the sense of not having a value in $\boldsymbol{\Psi}$.

References

[1] ACKERMANN, W. Grundgedanken einer typenfreien Logik. In [3] (1961), pp. 143–155.

[2] ACZEL, P., AND FEFERMAN, S. Consistency of the unrestricted abstraction principle using an intensional equivalence operator. In [14] (1980), pp. 67–98.

[3] BAR-HILLEL, Y., POZNANSKI, E. I. J., RABIN, M. O., AND ROBINSON, A., Eds. *Essays on the Foundations of Mathematics. Festschrift für A. Fraenkel.* Magnes Press, Jerusalem, 1961.

[4] Benacerraf, P., and Putnam, H., Eds. *Philosophy of Mathematics: Selected Readings*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[5] Cantini, A. The undecidability of Grišin's set theory. *Studia Logica 74*, 3 (2003), 345–368.

[6] Feferman, S. Toward useful type-free theories, I. *The Journal of Symbolic Logic 49* (1984), 75–111. Reprinted in [18], 237–287.

[7] Girard, J.-Y. Towards a geometry of interaction. In *Categories in Computer Science and Logic: Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference, June 14–20, 1987, Boulder Colorado* (Amsterdam and New York and Oxford, 1989), J. W. Gray and A. Scedrov, Eds., vol. 92 of *Contemporary Mathematics*, American Mathematical Society, pp. 69–108.

[8] Girard, J.-Y. Light linear logic. In [17] (1995), pp. 145–176. Reprinted with an additional remark in *Information and Computation* **143** (1998), 175–204.

[9] Girard, J.-Y. Linear logic: Its syntax and semantics. In [10] (1995), pp. 1–42.

[10] Girard, J.-Y., Lafont, Y., and Regnier, L., Eds. *Advances in Linear Logic*, vol. 222 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. Proceedings of the Workshop on Linear Logic, Ithaca, New York, June 1993.

[11] Griffor, E. R., Ed. *Handbook of Computability Theory*. Studies in Logic and the Foundations of Mathematics. Elsevier, Amsterdam, 1999.

[12] Hilbert, D. Über das Unendliche. *Mathematische Annalen 95* (1926), 161–190. Address delivered in Münster on 4 June 1925 at a meeting organized by the Westphalian Mathematical Society to honor the memory of Weierstrass. Partly translated in [4], 134–151; fully translated in [27], 369–392.

[13] Hilbert, D., and Bernays, P. *Die Grundlagen der Mathematik*, second ed., vol. I. Springer-Verlag, Berlin, Heidelberg, New York 1968.

[14] Hindley, J. R., and Seldin, J. P. *To H. B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, London, 1980.

[15] Kleene, S. C. Extension of an effectively generated class of functions by enumeration. *Colloquium Mathematicum VI* (1958), 67–78.
[16] Lafont, Y. Soft linear logic and polynomial time. *Theoretical Computer Science 318* (2004), 163–180.
[17] Leivant, D., Ed. *Logic and Computational Complexity. International Workshop LCC '94, Indianapolis, IN, USA, October 13–16, 1994. Selected Papers*, vol. 960 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, Heidelberg, New York, 1995.
[18] Martin, R. L., Ed. *Recent Essays on Truth and the Liar Paradox.* Oxford University Press, Oxford and New York, 1984.
[19] McKinley, R. Soft linear set theory. *Journal of Logic and Algebraic Programming 76*, 2 (2008), 226–245.
[20] Petersen, U. Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica 64* (2000), 365–403.
[21] Petersen, U. *Diagonal Method and Dialectical Logic. Tools, Materials, and Groundworks for a Logical Foundation of Dialectic and Speculative Philosophy.* Der Andere Verlag, Osnabrück, 2002.
[22] Petersen, U. $\mathbf{L^iD}_\lambda^Z$ as a basis for **PRA**. *Archive for Mathematical Logic 42* (2003), 665–694.
[23] Petersen, U. Some additions and corrections to *Diagonal Method and Dialectical Logic*. *Dilemmata: Jahrbuch der ASFPG 1* (2006), 93–173.
[24] Petersen, U. Begründung der Metaphysik durch logische Analyse der Selbstbezüglichkeit. *Dilemmata: Jahrbuch der ASFPG 2* (2007), 77–169.
[25] Schwichtenberg, H. Classifying recursive functions. In [11] (1999), pp. 533–586.
[26] Shirahata, M. Fixpoint theorem in linear set theory. Unpublished manuscript. Available at `http://www.fbc.keio.ac.jp/~sirahata/Research/`, 1999.
[27] van Heijenoort, J. *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, Cambridge, MA, 1967.