Some Additions and Corrections to *Diagonal Method and Dialectical Logic*[1]

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The following additions are meant to indicate some of the directions my research has taken since the publication of [**15**].

1. Addition 124g. Interpreting Weakening in LB$^\circ$

The point of this addition is to show that sacrificing weakening does not restrict expressive power in the presence of unrestricted abstraction.

A central issue in the development of a speculative logic is the question of how far one gets without any structural rules. In this context I shall present an interpretation of the formalized theory $\mathbf{LD}_\lambda$ as presented in [**15**], p. 472, definition 41.22 (4) (essentially Gentzen's **LK** without contraction but equipped with unrestricted λ-abstraction) in its intuitionistic linear subsystem. The relevant point is that $\bot \to A$ is available due to the definition of $\bot$ by means of unrestricted abstraction. The result is not in any way surprising but it seems to me of interest in view of linear logic and also in view of my ambitions to build logic without any structural rules.

The principal approach goes back to [**10**], but [**7**] was to become more influential. The approach taken here is in character closer to [**17**], pp. 49 f, although it still differs from it, not only in that I use different primitive symbols. It should be clear, however, that the present approach is in no way original and that it can be extended to theories built on linear logic, *i.e.*, abandoning weakening is in character very similar to shifting to intuitionistic logic from classical logic: in both cases it is double negation which holds the key to the interpretation, in the sense that adding double negation yields classical logic.

[1] Item [**15**] in the references for this paper starting on p. 171.

I begin by providing the relevant definitions.

Definition 1.1. The formalized theory $\mathbf{LB}^{\circ}$ is obtained from the formalized theory $\mathbf{LB}^{+}$ introduced in [**15**], p. 1682, definition 124.6 (4), by dropping weakening.

Intuitive consideration 1.2. The notion $\bot$ of *falsum* provides for the deducibility of $\bot \Rightarrow A$ (122.46v in [**15**], p. 1663). This, in turn, provides for a substitution of weakening: instead of $A \to (B \to A)$ the following is $\mathbf{LB}^{\circ}$-deducible:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \bot \Rightarrow \neg B}{\neg A, A \Rightarrow \neg B}}{\neg\neg B, \neg A, A \Rightarrow \bot}}{B, \neg A \Rightarrow \neg A} \qquad \bot \Rightarrow \bot}{\neg\neg A, \neg\neg B, \neg A \Rightarrow \bot}}{\neg\neg A, \neg\neg B \Rightarrow \neg\neg A}$$

Obviously $A \Rightarrow \neg\neg A$ is $\mathbf{LB}^{\circ}$-deducible. If double negation $\neg\neg A \Rightarrow A$ were also available, then this would be sufficient to prove weakening in the form $A \to (B \to A)$:

$$\begin{array}{c}\dfrac{\dfrac{B \Rightarrow \neg\neg B \qquad \dfrac{A \Rightarrow \neg\neg A \qquad \neg\neg A, \neg\neg B \Rightarrow \neg\neg A}{A, \neg\neg B \Rightarrow \neg\neg A}}{A, B \Rightarrow \neg\neg A} \qquad \neg\neg A \Rightarrow A}{A, B \Rightarrow A} \\ \hline\hline \Rightarrow A \to (B \to A)\end{array}$$

Apparently, however, weakening right is needed in a $\mathbf{LB}^{\circ}$-deduction of double negation:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, \bot}}{\Rightarrow A, A \to \bot} \qquad \bot \Rightarrow \bot}{(A \to \bot) \to \bot \Rightarrow A}$$

This is why I make recourse to the kind of interpretation that Gödel employed for the purpose of interpreting classical logic within intuitionistic logic.

PROPOSITION 1.3. *Inferences according to the following schemata are* **LB**°*-derivable.*

(1.3i) $$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \neg\neg A}$$

(1.3ii) $$\frac{\Gamma \Rightarrow \neg\neg A}{\Gamma, \neg A \Rightarrow C}$$

(1.3iii) $$\frac{A, \Gamma \Rightarrow \bot}{\neg\neg A, \Gamma \Rightarrow C}$$

(1.3iv) $$\frac{\neg\neg A, \Gamma \Rightarrow \bot}{A, \Gamma \Rightarrow \bot}$$

Proof. Straightforward. I only show 1.3ii as an example. Employ 122.46v feom [**15**], p. 1663:

$$\frac{\Gamma \Rightarrow \neg\neg A \qquad \dfrac{\neg A \Rightarrow \neg A \qquad \bot \Rightarrow C}{\neg A, \neg\neg A \Rightarrow C}}{\Gamma, \neg A \Rightarrow C}\clubsuit$$

QED

DEFINITION 1.4. $\|X\|$ is defined inductively as follows:

(1) $\|u\| :\equiv u$, u being a free or bound variable;
(2) $\|s \sqsubseteq t\| :\equiv \neg\neg(\|s\| \sqsubseteq \|t\|)$;
(3) $\|\lambda x\, \mathfrak{F}[x]\| :\equiv \lambda x \|\mathfrak{F}[x]\|$;
(4) If Γ ist the sequence $A_1, \ldots, A_\mathrm{m}$, then $\|\Gamma\|$ is the sequent $\|A_1\|, \ldots, \|A_\mathrm{m}\|$;
(5) $\|\Gamma \Rightarrow C\| :\equiv \|\Gamma\| \Rightarrow \|C\|$.

PROPOSITION 1.5. $\|C\|$ *has the form* $\neg\neg A$.

Proof. This is an obvious consequence of clause (2) of the foregoing definition in view of the fact that the outermost symbol of every wff in the language of **LB**° is $\sqsubseteq$: If $C \equiv s \sqsubseteq t$, then $\|C\| \equiv \neg\neg(\|s\| \sqsubseteq \|t\|)$. QED

Proposition 1.6. *Sequents according to the following schemata are* $\mathbf{LB}^{\circ}$*-deducible.*

(1.6i)	$\|\bot\| \Rightarrow \bot$
(1.6ii)	$\neg\neg\|A\| \Rightarrow \|A\|$
(1.6iii)	$\|(A \to \bot) \to \bot\| \Rightarrow \|A\|$
(1.6iv)	$\|\bot\|, \neg B \Rightarrow \bot$
(1.6v)	$\|A\|, \|B\| \Rightarrow \|A\|$
(1.6vi)	$\neg\neg(s \in \|b\|) \Rightarrow s \in \|b\|$
(1.6vii)	$\neg\neg(s \in \lambda x\, \|\mathfrak{A}[x]\|) \Rightarrow s \in \lambda x\, \|\mathfrak{A}[x]\|$

Proof. *Re* 1.6i.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \lambda\bot \sqsubseteq \lambda\bot}}{\Rightarrow \neg\neg(\lambda\bot \sqsubseteq \lambda\bot)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \sqsubseteq a \Rightarrow a \sqsubseteq a}{\Rightarrow \lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))}}{\Rightarrow \neg\neg(\lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))} \qquad \bot \Rightarrow \bot}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot) \Rightarrow \bot}}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda y\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x) \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x)) \Rightarrow \bot}}{\|\mathcal{V} \sqsubseteq \lambda x\,(\mathcal{V} \sqsubseteq x)\| \Rightarrow \bot}$$

Re 1.6ii. Let $\neg\neg\, A_1 \equiv \|A\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\|A\| \Rightarrow \|A\|}{\|A\| \Rightarrow \neg\neg A_1} \qquad \dfrac{\neg A_1 \Rightarrow \neg A_1 \qquad \bot \Rightarrow \bot}{\neg\neg A_1, \neg A_1 \Rightarrow \bot}}{\|A\|, \neg A_1 \Rightarrow \bot}}{\neg\neg\|A\|, \neg A_1 \Rightarrow \bot}}{\neg\neg\|A\| \Rightarrow \neg\neg A_1}}{\neg\neg\|A\| \Rightarrow \|A\|}$$

Re 1.6iii. Let $\neg\neg\, A_1 \equiv \|A\|$ according to proposition 1.5. Employ 122.46v from [**15**], p. 1663, and 1.6i:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A_1 \Rightarrow \neg A_1 \qquad \bot \Rightarrow \|\bot\|}{\neg A_1, \neg\neg A_1 \Rightarrow \|\bot\|}}{\neg A_1, \|A\| \Rightarrow \|\bot\|}}{\neg A_1 \Rightarrow \lambda\|A\| \sqsubseteq \lambda\|\bot\|}}{\neg A_1 \Rightarrow \neg\neg(\lambda\|A\| \sqsubseteq \lambda\|\bot\|)}}{\neg A_1 \Rightarrow \|(A \to \bot)\|} \qquad \|\bot\| \Rightarrow \bot}{\neg A_1, \lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\| \Rightarrow \bot}}{\neg A_1 \Rightarrow \neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|)} \qquad \bot \Rightarrow \bot}{\neg\neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|), \neg A_1 \Rightarrow \bot}}{\neg\neg(\lambda\|(A \to \bot)\| \sqsubseteq \lambda\|\bot\|) \Rightarrow \neg\neg A_1}}{\|(A \to \bot) \to \bot\| \Rightarrow \|A\|}
$$

Re 1.6iv. Employ 122.46v from [**15**], p. 1663:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \lambda\bot \sqsubseteq \lambda\bot}}{\Rightarrow \neg\neg(\lambda\bot \sqsubseteq \lambda\bot)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \sqsubseteq a \Rightarrow a \sqsubseteq a}{\Rightarrow \lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x)}}{\Rightarrow \neg\neg(\lambda x\,(x \sqsubseteq x) \sqsubseteq \lambda x\,(x \sqsubseteq x))} \qquad \dfrac{\bot \Rightarrow B \qquad \bot \Rightarrow \bot}{\bot, \neg B \Rightarrow \bot}}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot, \neg B \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda\bot), \neg B \Rightarrow \bot}}{\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x), \neg B \Rightarrow \bot}}{\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq \lambda x\,\neg\neg(\lambda y\,\neg\neg(y \sqsubseteq y) \sqsubseteq x)), \neg B \Rightarrow \bot}}{\|\bot\|, \neg B \Rightarrow \bot}
$$

Re 1.6v. Let $\neg\neg\, A_1 \equiv \|A\|$ and $\neg\neg\, B_1 \equiv \|B\|$ according to proposition 1.5. Employ 122.46v from [**15**], p. 1663:

$$\begin{array}{c}
A_1 \Rightarrow A_1 \qquad \bot \Rightarrow \neg B_1 \\ \hline
\neg A_1, A_1 \Rightarrow \neg B_1 \\ \hline
\neg\neg B_1, \neg A_1, A_1 \Rightarrow \bot \\ \hline
\neg\neg B_1, \neg A_1 \Rightarrow \neg A_1 \qquad \bot \Rightarrow \bot \\ \hline
\neg\neg A_1, \neg\neg B_1, \neg A_1 \Rightarrow \bot \\ \hline
\neg\neg A_1, \neg\neg B_1 \Rightarrow \neg\neg A_1 \\ \hline
\|A\|, \|B\| \Rightarrow \|A\|
\end{array}$$

Re 1.6vi.

$$\begin{array}{cl}
\neg(s\in b) \Rightarrow \neg(s\in b) \qquad \bot \Rightarrow \bot & \\ \hline
\neg\neg(s\in b), \neg(s\in b) \Rightarrow \bot & \\ \hline
s\in\lambda x\, \neg\neg(x\in b), \neg(s\in b) \Rightarrow \bot & \\ \hline
\neg\neg(s\in\lambda x\, \neg\neg(x\in b)), \neg(s\in b) \Rightarrow \bot & 1.3\text{iii} \\ \hline
\neg\neg(s\in\lambda x\, \neg\neg(x\in b)) \Rightarrow \neg\neg(s\in b) & \\ \hline
\neg\neg(s\in\lambda x\, \neg\neg(x\in b)) \Rightarrow s\in\lambda x\, \neg\neg(x\in b) &
\end{array}$$

Re 1.6vii. Let $\neg\neg\,\mathfrak{A}_1[s] \equiv \|\mathfrak{A}[s]\|$ according to proposition 1.5.

$$\begin{array}{cl}
\neg\mathfrak{A}_1[s] \Rightarrow \neg\mathfrak{A}_1[s] & \\ \hline
\neg\neg\mathfrak{A}_1[s], \neg\mathfrak{A}_1[s] \Rightarrow \bot & \\ \hline
\|\mathfrak{A}[s]\|, \neg\mathfrak{A}_1[s] \Rightarrow \bot & \\ \hline
s\in\lambda x\, \|\mathfrak{A}[x]\|, \neg\,\mathfrak{A}_1[s] \Rightarrow \bot & \\ \hline
\neg\neg(s\in\lambda x\, \|\mathfrak{A}[x]\|), \neg\,\mathfrak{A}_1[s] \Rightarrow \bot & 1.3\text{iii} \\ \hline
\neg\neg(s\in\lambda x\, \|\mathfrak{A}[x]\|) \Rightarrow \neg\neg\,\mathfrak{A}_1[s] & \\ \hline
\neg\neg(s\in\lambda x\, \|\mathfrak{A}[x]\|) \Rightarrow \|\mathfrak{A}[s]\| & \\ \hline
\neg\neg(s\in\lambda x\, \|\mathfrak{A}[x]\|) \Rightarrow s\in\lambda x\, \|\mathfrak{A}[x]\| &
\end{array}$$

QED

PROPOSITION 1.7. *Inferences according to the following schemata are* **LB**°*-derivable.*

(1.7i)
$$\frac{\Gamma \Rightarrow \neg\neg(s\in\|t\|)}{\Gamma \Rightarrow s\in\|t\|}$$

(1.7ii) $$\frac{s\in\|t\|, \Gamma \Rightarrow C}{\neg\neg(s\in\|t\|), \Gamma \Rightarrow C}$$

Proof. This are fairly immediate consequence of 1.6vi and 1.6vii. QED

The next step is to show that the interpretation of every $\mathbf{LD}_\lambda$-derivable inference is $\mathbf{LB}^\circ$-derivable.

PROPOSITION 1.8. *Inferences according to the following schemata are* $\mathbf{LB}^\circ$*-derivable.*

(1.8i) $$\frac{\|\Gamma \Rightarrow C\|}{\|A, \Gamma \Rightarrow C\|}$$

(1.8ii) $$\frac{\|\Gamma, A, B, \Pi \Rightarrow C\|}{\|\Gamma, B, A, \Pi \Rightarrow C\|}$$

(1.8iii) $$\frac{\|\Gamma \Rightarrow A\| \qquad \|A, \Pi \Rightarrow C\|}{\|\Gamma, \Pi \Rightarrow C\|}$$

(1.8iv) $$\frac{\|\Gamma \Rightarrow \mathfrak{A}[s]\| \qquad \|\mathfrak{B}[s], \Pi \Rightarrow C\|}{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \Gamma, \Pi \Rightarrow C\|}$$

(1.8v) $$\frac{\|\Gamma, \mathfrak{A}[a] \Rightarrow \mathfrak{B}[a]\|}{\|\Gamma \Rightarrow \lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y]\|}$$

Proof. Re 1.8i. This is 'weakening'. Employ 1.6iv. Distinguish two cases: empty antecedent or not. In the first case, let $\neg\neg B_1 \equiv \|B\|$ and $\neg\neg C_1 \equiv \|C\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\Rightarrow \|C\|}{\Rightarrow \neg\neg C_1} \qquad \dfrac{\neg C_1 \Rightarrow \neg C_1 \qquad \bot \Rightarrow \neg B_1}{\neg C_1, \neg\neg C_1 \Rightarrow \neg B_1}}{\dfrac{\dfrac{\dfrac{\neg C_1 \Rightarrow \neg B_1}{\neg\neg B_1 \Rightarrow \neg\neg C_1}}{\|B\| \Rightarrow \|C\|}}{\|B \Rightarrow C\|}}$$

In the second case, let $\mathit{\Gamma}$ be the sequence $A_1, \dots, A_{\mathrm{m}}$. Employ 1.6iv.

$$\dfrac{\dfrac{\|A\|, \|A_1\| \Rightarrow \|A_1\| \qquad \dfrac{\|A_1, \dots, A_m \Rightarrow C\|}{\|A_1\|, \dots, \|A_m\| \Rightarrow \|C\|}}{\|A\|, \|A_1\|, \dots, \|A_m\| \Rightarrow \|C\|}\,\clubsuit}{\|A, A_1, \dots, A_m \Rightarrow C\|}$$

Re 1.8ii. This is ‘exchange’. Obvious. left to the reader.
Re 1.8iii. This is ‘cut’.

$$\dfrac{\dfrac{\dfrac{\dfrac{\|\mathit{\Gamma} \Rightarrow A\|}{\|\mathit{\Gamma}\| \Rightarrow \|A\|} \qquad \dfrac{\|A, \mathit{\Pi} \Rightarrow C\|}{\|A\|, \|\mathit{\Pi}\| \Rightarrow \|C\|}}{\|\mathit{\Gamma}\|, \|\mathit{\Pi}\| \Rightarrow \|C\|}\,\clubsuit}{\|\mathit{\Gamma}\|, \|\mathit{\Pi}\| \Rightarrow \|C\|}}{\|\mathit{\Gamma}, \mathit{\Pi} \Rightarrow C\|}$$

Re 1.8iv. This is $\sqsubseteq$-left rule. Let $\neg\neg\, C_1 \equiv \|C\|$ according to proposition 1.5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\|\mathit{\Gamma} \Rightarrow \mathfrak{A}[s]\|}{\|\mathit{\Gamma}\| \Rightarrow \|\mathfrak{A}[s]\|} \qquad \dfrac{\dfrac{\|\mathfrak{B}[s], \mathit{\Pi} \Rightarrow C\|}{\|\mathfrak{B}[s]\|, \|\mathit{\Pi}\| \Rightarrow \|C\|}}{\|\mathfrak{B}[s]\|, \|\mathit{\Pi}\| \Rightarrow \neg\neg C_1}}{\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|, \|\mathit{\Gamma}\|, \|\mathit{\Pi}\| \Rightarrow \neg\neg C_1}}{\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|, \|\mathit{\Gamma}\|, \|\mathit{\Pi}\|, \neg C_1 \Rightarrow \bot}}{\neg\neg(\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|), \|\mathit{\Gamma}\|, \|\mathit{\Pi}\|, \neg C_1 \Rightarrow \bot}\,\text{1.3iii}}{\neg\neg(\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|), \|\mathit{\Gamma}\|, \|\mathit{\Pi}\| \Rightarrow \neg\neg C_1}}{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \|\mathit{\Gamma}\|, \|B\| \Rightarrow \|C\|}}{\|\lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y], \mathit{\Gamma}, \mathit{\Pi} \Rightarrow C\|}$$

Re 1.8v. This is ‘$\sqsubseteq$-right’.

$$\dfrac{\dfrac{\dfrac{\dfrac{\|\Gamma, \mathfrak{A}[a] \Rightarrow \mathfrak{B}[a]\|}{\|\Gamma\|, \|\mathfrak{A}[a]\| \Rightarrow \|\mathfrak{B}[a]\|}}{\|\Gamma\| \Rightarrow \lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|}}{\|\Gamma\| \Rightarrow \neg\neg(\lambda x\, \|\mathfrak{A}[x]\| \sqsubseteq \lambda y\, \|\mathfrak{B}[y]\|)}}{\|\Gamma \Rightarrow \lambda x\, \mathfrak{A}[x] \sqsubseteq \lambda y\, \mathfrak{B}[y]\|}$$

QED

2. Addition 130d. Application of the fixed point property: a numeralwise representation of the recursive functions in $\mathbf{L^iD_\lambda}$[2]

The possibility of obtaining a definition of the natural numbers in $\mathbf{L^iD_\lambda}$ that would provide induction in a "second order style" as, *e.g.*, in section 41f in [**15**], is out of question for simple ordinal reasons: the consistency of $\mathbf{L^iD_\lambda}$ is already provable by means of a simple induction. As a result, the possibility of defining recursive functions in a "Dedekind style" is not open.[3]

There is, however, the possibility of numeralwise representing all recursive functions. This possibility is essentially based on two features of contraction free logic with unrestricted abstraction, *viz.*,

— the (direct) fixed point property, and
— the contractibility of $\equiv$-wffs.

The (direct) fixed point property provides for terms that numeralwise represent recursive functions somewhat like the recursion theorem provides for partial recursive functions.[4] What is specific about this numeralwise representation of recursive function is the role of identity; *i.e.*, what is

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[2] This addition was sparked by [**18**] and [**19**]. Cf. also [**6**]. An actual proof of the numeralwise representability of the recursive functions does not seem to be available in print. [**18**] is not published and [**19**] only states the result with reference to [**18**].

[3] It is possible, of course, to provide definitions in that style, but due to the deductive weakness of $\mathbf{L^iD_\lambda}$ their characteristic properties cannot be proved in $\mathbf{L^iD_\lambda}$. As emphasized in [**19**], p. 10 (albeit with regard to a slightly different system), "such a theory is descriptively rich" but "proof theoretically very weak (as its consistency is established by the induction up to ω)."

[4] There is a significant difference, though: the recursion theorem is compatible with classical logic, but not so the (direct) fixed point theorem.

being considered are numerals, not anything that equals it.[5] In this way some valuable classical features are rescued for our non-classical situation like the very contractibility of $\equiv$-wffs.

This approach works well for all functions defined by n-recursion. It is the sort of closure operation constituted by minimization that needs special attention. What is required is a form of trichotomy in order to prove that minimization can be numeralwise represented.

The proximity of the proof presented here to the one in [**3**], pp. 192–199, or [**2**], pp. 166–171, for the case of Robinson's arithmetic will be obvious. The main point is that the smaller relation and with it the representation of the least number operater is based on a term $\mathbf{B}^*$ which is introduced as a fixed point. It acts like a strengthened kind of B-axiom[6] in that it allows to prove a form of trichotomy. As in the case of Robinson's arithmetic heavy weight lies on the use of meta-theoretical induction. That's where results are only established for numerals.

I begin with an adaptation of the notion of *numeralwise representation* to the situation of $\mathbf{L^iD}_\lambda$.

DEFINITIONS 2.1. (1) A k-place total function $\mathfrak{f}$ is said to be numeralwise represented by f in $\mathbf{L^iD}_\lambda$, if the following holds:

$$\text{if } \mathfrak{f}(\vec{\mathrm{n}}) = \mathrm{m}, \text{ then } \begin{cases} \vdash_{\mathbf{L^iD}_\lambda} \langle\!\langle \vec{n}, m \rangle\!\rangle \in f \\ \vdash_{\mathbf{L^iD}_\lambda} \bigwedge x (\langle\!\langle \vec{n}, x \rangle\!\rangle \in f \to x \equiv m) \end{cases}$$

for all k-tuples $\vec{\mathrm{n}}$ of natural numbers and natural numbers m.
(2) A function $\mathfrak{f}$ is said to be *numeralwise representable in* $\mathbf{L^iD}_\lambda$, if there is a term t which numeralwise represents $\mathfrak{f}$ in $\mathbf{L^iD}_\lambda$.

Next come the exclusive successor notion and some of its properties which will be needed later.

DEFINITION 2.2. $s^{\hat{\imath}} :\equiv \lambda x (x \in s \diamond x \equiv s)$.

[5] This means, in particular, that functions cannot be employed to apply to arguments; *i.e.*, , instead of $f[\![x]\!] = y$ one only has something like $\langle\!\langle y, x \rangle\!\rangle \in f$.

[6] Cf. in definition 128.36 on p. 1764 of [**15**].

Proposition 2.3. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.3i) $\Rightarrow s \in s^{\prime}$

(2.3ii) $s^{\prime} \equiv 0 \Rightarrow$

(2.3iii) $s^{\prime} \equiv t^{\prime} \Rightarrow s \in t^{\prime}$

(2.3iv) $s \in n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n$

(2.3v) $s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n$

(2.3vi) $s \equiv n^{\prime}, s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \Rightarrow$

(2.3vii) $s^{\prime} \equiv n^{\prime} \Rightarrow s \equiv n$

Proof. *Re* 2.3i and 2.3ii. As for their inclusive counterparts, cf. 128.29i and 128.29ii in [**15**], p. 1759.
Re 2.3iii.

$$\begin{array}{c} \Rightarrow s \in s^{\prime} \qquad s \in t^{\prime} \Rightarrow s \in t^{\prime} \\ \hline s \in s^{\prime} \to s \in t^{\prime} \Rightarrow s \in t^{\prime} \\ \hline s^{\prime} \equiv t^{\prime} \Rightarrow s \in t^{\prime} \end{array}$$

Re 2.3iv. Employ a meta-theoretical induction on n.
n = 0:

$$\begin{array}{c} s \in 0 \Rightarrow \qquad s \equiv 0 \Rightarrow s^{\prime} \equiv 0^{\prime} \\ \hline s \in 0^{\prime} \Rightarrow s \equiv 0 \\ \hline s \in 0^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \\ \hline\hline s \in 0^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \end{array}$$

n = $m^{\prime}$:

$$\begin{array}{c} s \in m^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \qquad s \equiv m^{\prime} \Rightarrow s \equiv m^{\prime} \\ \hline s \in m^{\prime} \diamond s \equiv m^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \diamond s \equiv m^{\prime} \\ \hline s \in m^{\prime\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv m \diamond s \equiv m^{\prime} \end{array}$$

Re 2.3v. Employ a cut on 2.3iii and 2.3iv:

$$\begin{array}{c} s^{\prime} \equiv n^{\prime} \Rightarrow s \in n^{\prime} \qquad s \in n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \\ \hline s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \end{array} \clubsuit$$

Re 2.3vi. Employ a meta-theoretical induction on n. For n = 0, the situation is immediately clear from 126.45i in [**15**]:

As regards $\mathrm{n} = \mathrm{m}^{\prime}$:

$$\dfrac{\dfrac{\dfrac{m^{\prime\prime} \equiv 0^{\prime} \Rightarrow}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv 0^{\prime} \Rightarrow} \quad \ldots \quad \dfrac{m^{\prime\prime} \equiv m^{\prime} \Rightarrow}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv m^{\prime} \Rightarrow}}{\text{n } \diamond\text{-introductions left}}}{s^{\prime} \equiv m^{\prime\prime}, s^{\prime} \equiv 0 \diamond \cdots \diamond s^{\prime} \equiv m^{\prime} \Rightarrow}$$

Re 2.3vii. This is now an immediate consequence of 2.3v and 2.3vi:

$$\dfrac{s^{\prime} \equiv n^{\prime} \Rightarrow s^{\prime} \equiv 0^{\prime} \diamond \cdots \diamond s^{\prime} \equiv n \diamond s \equiv n \qquad s^{\prime} \equiv n^{\prime}, s^{\prime} \equiv 0 \diamond \cdots \diamond s^{\prime} \equiv n \Rightarrow}{s^{\prime} \equiv n^{\prime} \Rightarrow s \equiv n}$$

QED

In view of result 10.6 in [**15**], p. 77, it is sufficient to consider:

1. basic functions **Z**, **S**, **I**, and the characteristic function of equality
2. composition
3. addition and multiplication
4. minimization

I begin with a numeralwise representation of the functions listed under 1 and 2.

DEFINITIONS 2.4. (1) $\mathit{zero} :\equiv \lambda xy(y \equiv 0)$.
(2) $\mathit{suc} :\equiv \lambda xy(y \equiv x^{\prime})$.
(3) $\mathit{id}^{\mathrm{m}}_{\mathrm{n}} :\equiv \lambda \vec{x} y(y \equiv x_{\mathrm{n}})$.
(4) $\mathit{char}_{=} :\equiv \lambda xyz((x \equiv y \sqcap z \equiv 0) \vee (x \not\equiv y \sqcap z \equiv 1))$.
(4) $\mathit{comp}[h, \vec{g}] :\equiv \lambda \vec{x} \vec{y} z(\langle \vec{x}, y_1 \rangle \in g_1 \sqcap \ldots \sqcap \langle \vec{x}, y_{\mathrm{n}} \rangle \in g_{\mathrm{n}} \sqcap \langle \vec{y}, z \rangle \in h)$.

REMARK 2.5. In view of the definition of $\lambda xy\, \mathfrak{F}[x, y]$, the definition of *zero*, *e.g.*, amounts to $\lambda z \bigvee x \bigvee y(z \equiv \langle x, y \rangle \sqcap y \equiv 0)$.[7]

PROPOSITION 2.6.

(2.6i) *zero numeralwise represents the zero function* **Z**

(2.6ii) *suc numeralwise represents the successor function* **S**

(2.6iii) *id numeralwise represents* **I**

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[7] The axiom employed in [**18**] amounts to $\lambda z \bigvee x(z \equiv \langle x, 0 \rangle)$ in my symbolism.

(2.6iv) *char*$_=$ *numeralwise represents the characteristic function of equality* χ_{eq}

(2.6v) $comp[h, \vec{g}]$ *numeralwise represents the composition of functions* $\mathrm{Cn}[h, g_1, \ldots, g_\mathrm{m}]$

Proof. Completely straightforward, but to see the point of the notion of identity in the definitions, I just indicate how to treat the case of *zero*: What has to be shown is

$$\vdash_{\mathbf{L^iD}_\lambda} \langle n, 0\rangle \in \lambda xy\,(y \equiv 0), \text{ and}$$
$$\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle n, x\rangle \in zero \to x \equiv 0)\,.$$

The first one reduces to $0 \equiv 0$ and the second one to $a \equiv 0 \Rightarrow a \equiv 0$. QED

Proposition 2.7. *There are terms add and mult satisfying*

(2.7i) $\mathbf{L^iD}_\lambda \vdash \; add = \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv x_1)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{i} \,\square\, x_3 \equiv z^{i} \,\square\, \langle\langle x_1, y\rangle, z\rangle \in add))$

(2.7ii) $\mathbf{L^iD}_\lambda \vdash \; mult = \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv 0)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{i} \,\square\, \langle\langle z, x_1\rangle, x_3\rangle \in add \,\square\, \langle\langle x_1, y\rangle, z\rangle \in mult))$

Proof. This is again an immediate consequence of the fixed point property. QED

The following convention is introduced for the convenience of formulating results regarding *add* and *mult*.

Convention 2.8.

(1) $\mathcal{ADD} :\equiv \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv x_1)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{i} \,\square\, x_3 \equiv z^{i} \,\square\, \langle\langle x_1, y\rangle, z\rangle \in add))$

(2) $\mathcal{MULT} :\equiv \lambda x_1 x_2 x_3\,((x_2 \equiv 0 \,\square\, x_3 \equiv 0)\diamond \bigvee y \bigvee z\,(x_2 \equiv y^{i} \,\square\, \langle\langle z, x_1\rangle, x_3\rangle \in add \,\square\, \langle\langle x_1, y\rangle, z\rangle \in mult))$

COROLLARY 2.9. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable*

$$\text{(2.9i)} \qquad \frac{s \in \mathcal{ADD}, \Gamma \Rightarrow C}{s \in add, \Gamma \Rightarrow C}$$

$$\text{(2.9ii)} \qquad \frac{\Gamma \Rightarrow s \in \mathcal{ADD}}{\Gamma \Rightarrow s \in add}$$

$$\text{(2.9iii)} \qquad \frac{s \in \mathcal{MULT}, \Gamma \Rightarrow C}{s \in mult, \Gamma \Rightarrow C}$$

$$\text{(2.9iv)} \qquad \frac{\Gamma \Rightarrow s \in \mathcal{MULT}}{\Gamma \Rightarrow s \in mult}$$

PROPOSITION 2.10. *Sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.10i) $\Rightarrow \langle\langle s, 0\rangle, s\rangle \in add$

(2.10ii) $\langle\langle s, t\rangle, r\rangle \in add \Rightarrow \langle\langle s, t'\rangle, r'\rangle \in add$

(2.10iii) $\langle\langle s, 0\rangle, t\rangle \in add \Rightarrow t \equiv s$

(2.10iv) $\bigwedge x (\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p), \langle\langle s, n'\rangle, t\rangle \in add \Rightarrow t \equiv p'$

Proof. Re 2.10i.

$$\dfrac{\dfrac{\dfrac{\Rightarrow 0 \equiv 0 \qquad \Rightarrow s \equiv s}{\Rightarrow 0 \equiv 0 \,\square\, s \equiv s}}{\Rightarrow (0 \equiv 0 \,\square\, s \equiv s) \diamond \bigvee y \bigvee z\, (0 \equiv y' \,\square\, s \equiv z' \,\square\, \langle\langle s, y\rangle, z\rangle \in add)}}{\Rightarrow \langle\langle s, 0\rangle, s\rangle \in add}$$

Re 2.10ii. In view of 2.19i below, this is left to the reader.

Re 2.10iii.

$$\dfrac{\dfrac{\dfrac{t \equiv s \Rightarrow t \equiv s}{0 \equiv 0, t \equiv s \Rightarrow t \equiv s}}{0 \equiv 0 \,\square\, t \equiv s \Rightarrow t \equiv s} \qquad \dfrac{\dfrac{\dfrac{0 \equiv b^{\prime} \Rightarrow}{0 \equiv b^{\prime}, t \equiv c^{\prime}, \langle\langle s, b\rangle, c\rangle \in add) \Rightarrow}}{0 \equiv b^{\prime} \,\square\, t \equiv c^{\prime} \,\square\, \langle\langle s, b\rangle, c\rangle \in add) \Rightarrow}}{\bigvee y \bigvee z\,(0 \equiv y^{\prime} \,\square\, t \equiv z^{\prime} \,\square\, \langle\langle s, y\rangle, z\rangle \in add) \Rightarrow}}{\dfrac{(0 \equiv 0 \,\square\, t \equiv s) \diamond \bigvee y \bigvee z\,(0 \equiv y^{\prime} \,\square\, t \equiv z^{\prime} \,\square\, \langle\langle s, y\rangle, z\rangle \in add) \Rightarrow t \equiv s}{\langle\langle s, 0\rangle, t\rangle \in add \Rightarrow t \equiv s}\; 2.9\text{ii}}$$

Re 2.10iv. Let $\mathcal{A}$ stand for $\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p)$ and $\mathcal{C}$ for $\bigvee y \bigvee z\,(n^{\prime} \equiv y^{\prime} \,\square\, t \equiv z^{\prime} \,\square\, \langle\langle s, y\rangle, z\rangle \in add)$:

$$\dfrac{\Rightarrow \neg(n^{\prime} \equiv 0 \,\square\, t \equiv s) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle\langle s, n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n\rangle, c\rangle \in add \qquad \dfrac{c \equiv p \Rightarrow c^{\prime} \equiv p^{\prime}}{c \equiv p, t \equiv c^{\prime} \Rightarrow t \equiv p^{\prime}}}{\langle\langle s, n\rangle, c\rangle \in add \rightarrow c \equiv p, t \equiv c^{\prime}, \langle\langle s, n\rangle, c\rangle \in add \Rightarrow t \equiv p^{\prime}}}{\mathcal{A}, t \equiv c^{\prime}, \langle\langle s, n\rangle, c\rangle \in add \Rightarrow t \equiv p^{\prime}}}{\mathcal{A}, n \equiv b, t \equiv c^{\prime}, \langle\langle s, b\rangle, c\rangle \in add \Rightarrow t \equiv p^{\prime}}}{\mathcal{A}, n^{\prime} \equiv b^{\prime}, t \equiv c^{\prime}, \langle\langle s, b\rangle, c\rangle \in add \Rightarrow t \equiv p^{\prime}}\; 2.3\text{vii}}{\mathcal{A}, n^{\prime} \equiv b^{\prime} \,\square\, t \equiv c^{\prime} \,\square\, \langle\langle s, b\rangle, c\rangle \in add \Rightarrow t \equiv p^{\prime}}}{\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p), \mathcal{C} \Rightarrow t \equiv p^{\prime}}}{\dfrac{\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p), (n^{\prime} \equiv 0 \,\square\, t \equiv s) \diamond \mathcal{C} \Rightarrow t \equiv p^{\prime}}{\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in add \rightarrow x \equiv p), \langle\langle s, n^{\prime}\rangle, t\rangle \in add \Rightarrow t \equiv p^{\prime}}\; 2.9\text{ii}}$$

QED

PROPOSITION 2.11. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.11i) $\Rightarrow \langle\langle s, 0\rangle, 0\rangle \in mult$

(2.11ii) $\langle\langle s, t\rangle, s_1\rangle \in mult, \langle\langle s_1, s\rangle, r\rangle \in add \Rightarrow \langle\langle s, t^{\prime}\rangle, r\rangle \in mult$

(2.11iii) $\langle\langle s, 0\rangle, t\rangle \in mult \Rightarrow t \equiv 0$

(2.11iv) $\bigwedge x\,(\langle\langle s, n\rangle, x\rangle \in mult \rightarrow x \equiv r_1), \langle\langle s, n^{\prime}\rangle, t\rangle \in mult,$
$\bigwedge x\,(\langle\langle r_1, s\rangle, x\rangle \in add \rightarrow x \equiv r_2) \Rightarrow t \equiv r_2$

Proof. Essentially as for 2.10; I shall only treat the second as an example. *Re* 2.11ii. To save space, let $\mathcal{C}$ be short for $(t^{\prime} \equiv 0 \,\Box\, r \equiv 0)$ and $\mathcal{A}$ for $\langle\langle s, t\rangle, s_1\rangle \in \mathit{mult}$:

$$
\begin{array}{c}
\mathcal{A} \Rightarrow t^{\prime} \equiv t^{\prime} \,\Box\, \langle\langle s, t\rangle, s_1\rangle \in \mathit{mult} \qquad \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \Rightarrow \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \\
\hline\hline
\langle\langle s, t\rangle, s_1\rangle \in \mathit{mult}, \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \Rightarrow t^{\prime} \equiv t^{\prime} \,\Box\, \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \,\Box\, \langle\langle s, t\rangle, s_1\rangle \in \mathit{mult} \\
\hline\hline
\mathcal{A}, \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \Rightarrow \bigvee y \bigvee z\, (t^{\prime} \equiv y^{\prime} \,\Box\, \langle\langle z, s\rangle, r\rangle \in \mathit{add} \,\Box\, \langle\langle x_1, y\rangle, z\rangle \in \mathit{mult}) \\
\hline
\mathcal{A}, \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \Rightarrow \mathcal{C} \diamond \bigvee y \bigvee z\, (t^{\prime} \equiv y^{\prime} \,\Box\, \langle\langle z, s\rangle, r\rangle \in \mathit{add} \,\Box\, \langle\langle x_1, y\rangle, z\rangle \in \mathit{mult}) \\
\hline
\langle\langle s, t\rangle, s_1\rangle \in \mathit{mult}, \langle\langle s_1, s\rangle, r\rangle \in \mathit{add} \Rightarrow \langle\langle s, t^{\prime}\rangle, r\rangle \in \mathit{mult}
\end{array}
$$

Re 2.11iv. To save space, let $\mathcal{A}$ stand for $\bigwedge x\, (\langle\langle s, n\rangle, x\rangle \in \mathit{mult} \rightarrow x \equiv r_1)$ and $\mathcal{C}$ for $\bigvee y \bigvee z\, (n^{\prime} \equiv y^{\prime} \,\Box\, \langle\langle z, s\rangle, t\rangle \in \mathit{add} \,\Box\, \langle\langle s, y\rangle, z\rangle \in \mathit{mult})$ and $\mathcal{F}$ for $\bigwedge x\, (\langle\langle r_1, s\rangle, x\rangle \in \mathit{add} \rightarrow x \equiv r_2)$:

$$
\begin{array}{c}
\langle\langle r_1, s\rangle, t\rangle \in \mathit{add} \Rightarrow \langle\langle r_1, s\rangle, t\rangle \in \mathit{add} \qquad t \equiv r_2 \Rightarrow t \equiv r_2 \\
\hline
\langle\langle r_1, s\rangle, t\rangle \in \mathit{add} \rightarrow t \equiv r_2, \langle\langle r_1, s\rangle, t\rangle \in \mathit{add} \Rightarrow t \equiv r_2 \\
\hline
\mathcal{F}, \langle\langle r_1, s\rangle, t\rangle \in \mathit{add} \Rightarrow t \equiv r_2 \\
\hline
c \equiv r_1, \mathcal{F}, \langle\langle c, s\rangle, t\rangle \in \mathit{add} \Rightarrow t \equiv r_2
\end{array}
$$

$$
\begin{array}{c}
\langle\langle s, n\rangle, c\rangle \in \mathit{mult} \Rightarrow \langle\langle s, n\rangle, c\rangle \in \mathit{mult} \qquad c \equiv r_1, \mathcal{F}, \langle\langle c, s\rangle, t\rangle \in \mathit{add} \Rightarrow t \equiv r_2 \\
\hline
\langle\langle s, n\rangle, c\rangle \in \mathit{mult} \rightarrow c \equiv r_1, \langle\langle c, s\rangle, t\rangle \in \mathit{add}, \langle\langle s, n\rangle, c\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline
\mathcal{A}, \langle\langle c, s\rangle, t\rangle \in \mathit{add}, \langle\langle s, n\rangle, c\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline
\mathcal{A}, n \equiv b, \langle\langle c, s\rangle, t\rangle \in \mathit{add}, \langle\langle s, b\rangle, c\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline \text{2.3vii} \\
\mathcal{A}, n^{\prime} \equiv b^{\prime}, \langle\langle c, s\rangle, t\rangle \in \mathit{add}, \langle\langle s, b\rangle, c\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline\hline
\mathcal{A}, n^{\prime} \equiv b^{\prime} \,\Box\, \langle\langle c, s\rangle, t\rangle \in \mathit{add} \,\Box\, \langle\langle s, b\rangle, c\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline\hline
\mathcal{A}, \mathcal{C}, \mathcal{F} \Rightarrow t \equiv r_2
\end{array}
$$

$$
\begin{array}{c}
\Rightarrow \neg(n^{\prime} \equiv 0 \,\Box\, t \equiv 0) \qquad \mathcal{A}, \mathcal{C}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline
\mathcal{A}, \neg(n^{\prime} \equiv 0 \,\Box\, n \equiv 0) \diamond \mathcal{C}, \mathcal{F} \Rightarrow t \equiv r_2 \\
\hline \text{2.9iv} \\
\mathcal{A}, \langle\langle s, n^{\prime}\rangle, t\rangle \in \mathit{mult}, \mathcal{F} \Rightarrow t \equiv r_2
\end{array}
$$

QED

PROPOSITION 2.12.

(2.12i) *add numeralwise represents the function* $+$

(2.12ii) *mult numeralwise represents the function* $\cdot$

Proof. Re 2.12i. What has to be shown is that if m + n = p, then

$$\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,n\rangle,p\rangle\in add, \text{ and}$$
$$\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,n\rangle,x\rangle\in add \to x\equiv p)\,.$$

In both cases, employ a meta-theoretical induction on n.
As regards the first one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,0\rangle,m\rangle\in add$. This is 2.10i.
n = k′. What has to be shown is that if p is the numerical value of m + k, then $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k'\rangle,p'\rangle\in add$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k\rangle,p\rangle\in add$. This yields the claim by a cut with 2.10ii.
As regards the second one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,0\rangle,x\rangle\in add \to x\equiv m)$. This is easily obtained from 2.10iii.
n = k′. What has to be shown is that if p is the numerical value of m + k, then $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,k'\rangle,x\rangle\in add \to x\equiv p')$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,k\rangle,x\rangle\in add \to x\equiv p)$. By a cut with 2.10ii this yields $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k'\rangle,c\rangle\in add \Rightarrow c\equiv p'$ which yields the claim by →- and ⋀-introduction.
Re 2.12ii. What has to be shown is that if m · n = p, then

$$\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,n\rangle,p\rangle\in mult, \text{ and}$$
$$\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,n\rangle,x\rangle\in mult \to x\equiv p)\,.$$

Again, employ meta-theoretical inductions on n.
As regards the first one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,0\rangle,0\rangle\in mult$. This is 2.11i.
n = k′. What has to be shown is that if p is the numerical value of m · k and q is the numerical value of p + m, then $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k'\rangle,q\rangle\in mult$. By the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k\rangle,p\rangle\in mult$ and by 2.12i, $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle p,m\rangle,q\rangle\in add$. Two cuts with 2.11ii yield the claim.
As regards the second one:
n = 0. What has to be shown is $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,0\rangle,x\rangle\in mult \to x\equiv 0)$. This is easily obtained from 2.11iii.
n = k′. Let p be the numerical value of m · k and q that of p + m. Then, by the induction hypothesis, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,k\rangle,x\rangle\in mult \to x\equiv p)$ and by 2.12i, $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle p,m\rangle,x\rangle\in add \to x\equiv q)$. Two cuts with 2.11iv yield $\vdash_{\mathbf{L^iD}_\lambda} \langle\langle m,k'\rangle,t\rangle\in mult \Rightarrow t\equiv q$. Applying a →- and a ⋀-introduction then yields $\vdash_{\mathbf{L^iD}_\lambda} \bigwedge x\,(\langle\langle m,k'\rangle,x\rangle\in mult \to x\equiv q)$. QED

Remark 2.13. Of course, *all* total functions definable by 1-recursion can be numeralwise represented in that way. If the function $\mathfrak{f}$ is defined by primitive recursion from the functions $\mathfrak{g}$ and $\mathfrak{h}$, and $\mathfrak{g}$ and $\mathfrak{h}$ are represented in $\mathbf{L^iD}_\lambda$ by g and h, respectively, then $\mathfrak{f}$ is represented by the term f satisfying the following fixed point property in $\mathbf{L^iD}_\lambda$:

$$f = \lambda x_1 x_2 x_3\, ((x_2 \equiv 0 \sqcap \langle x_1, x_3\rangle \in g) \diamond \\ \bigvee y \bigvee z\, (x_2 \equiv y' \sqcap \langle\langle x_1, y\rangle, z\rangle \in f \sqcap \langle\langle\langle x_1, y\rangle, z\rangle, x_3\rangle \in h))\,.$$

As a matter of fact, n-recursion can be represented in that way too. As an example, consider the so-called *Ackermann function*. Employ the following fixed point *ak* for a numeralwise representation of the Ackermann function:

$$ak = \lambda x_1 x_2 x_3\, ((x_1 \equiv 0 \sqcap x_3 \equiv x_2{}') \diamond \\ \bigvee y\, (x_1 \equiv y' \sqcap x_2 \equiv 0 \sqcap \langle\langle y, 0'\rangle, x_3\rangle \in ak) \diamond \\ \bigvee y_1 \bigvee y_2 \bigvee z\, (x_1 \equiv y_1{}' \sqcap x_2 \equiv y_2{}' \sqcap \langle\langle x_1, y_2\rangle, z\rangle \in ak \sqcap \langle\langle x_1, z\rangle, x_3\rangle \in ak)\,.$$

In other words, all stages of recursion can be numeralwise represented in a straightforward manner. This may provoke the question as to what the least number operator actually adds to the notion of recursion.

The following schemata of inference will come handy in the further presentation. They are instances of what I called an "exclusion principle" in remarks 116.6 and 119.1 in [**15**], for example.

Proposition 2.14. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable.*

$$\text{(2.14i)} \quad \frac{\Gamma \Rightarrow \mathfrak{F}[s, 0, s] \qquad \Gamma, \langle\langle s, a\rangle, b\rangle \in add \Rightarrow \mathfrak{F}[s, a', b']}{\Gamma, \langle\langle s, t\rangle, r\rangle \in add \Rightarrow \mathfrak{F}[s, t, r]}$$

$$\text{(2.14ii)} \quad \frac{\Gamma \Rightarrow \mathfrak{F}[s, 0, s] \qquad \Gamma, \langle\langle b, s\rangle, r\rangle \in add, \langle\langle s, a\rangle, b\rangle \in mult \Rightarrow \mathfrak{F}[s, a', r]}{\Gamma, \langle\langle s, t\rangle, r\rangle \in mult \Rightarrow \mathfrak{F}[s, t, r]}$$

Proof. *Re* 2.14i.

$$
\dfrac{
\dfrac{\dfrac{\Gamma \Rightarrow \mathfrak{F}[s,0,s]}{\Gamma, t \equiv 0, r \equiv s \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, t \equiv 0 \sqcap r \equiv s \Rightarrow \mathfrak{F}[s,t,r]}
\quad
\dfrac{\dfrac{\dfrac{\Gamma, \langle\langle s,a\rangle, b\rangle \in add \Rightarrow \mathfrak{F}[s,a^{\prime},b^{\prime}]}{\Gamma, t \equiv a^{\prime}, r \equiv b^{\prime}, \langle\langle s,a\rangle, b\rangle \in add \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, t \equiv a^{\prime} \sqcap r \equiv b^{\prime} \sqcap \langle\langle s,a\rangle, b\rangle \in add \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, \bigvee y \bigvee z (t \equiv y^{\prime} \sqcap r \equiv z^{\prime} \sqcap \langle\langle s,y\rangle, z\rangle \in add) \Rightarrow \mathfrak{F}[s,t,r]}
}{
\dfrac{\Gamma, (t \equiv 0 \sqcap r \equiv s) \diamond \bigvee y \bigvee z\, (t \equiv y^{\prime} \sqcap r \equiv z^{\prime} \sqcap \langle\langle s,y\rangle, z\rangle \in add) \Rightarrow \mathfrak{F}[s,t,r]}{\Gamma, \langle\langle s,t\rangle, r\rangle \in add \Rightarrow \mathfrak{F}[s,t,r]}
}
$$

Re 2.14ii. Let $\mathfrak{A} :\equiv (t \equiv {*_1}^{\prime} \sqcap \langle\langle *_2, s\rangle, r\rangle \in add \sqcap \langle\langle s, *_1\rangle, *_2\rangle \in mult)$:

$$
\dfrac{
\dfrac{\dfrac{\Gamma \Rightarrow \mathfrak{F}[s,0,0]}{\Gamma, t \equiv 0, r \equiv 0 \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, t \equiv 0 \sqcap r \equiv 0 \Rightarrow \mathfrak{F}[s,t,r]}
\quad
\dfrac{\dfrac{\dfrac{\Gamma, \langle\langle b,s\rangle, r\rangle \in add, \langle\langle s,a\rangle, b\rangle \in mult \Rightarrow \mathfrak{F}[s,a^{\prime},r]}{\Gamma, t \equiv a^{\prime}, \langle\langle b,s\rangle, r\rangle \in add, \langle\langle s,a\rangle, b\rangle \in mult \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, \mathfrak{A}[a,b] \Rightarrow \mathfrak{F}[s,t,r]}}{\Gamma, \bigvee y \bigvee z\, \mathfrak{A}[y,z] \Rightarrow \mathfrak{F}[s,t,r]}
}{
\dfrac{\Gamma, (t \equiv 0 \sqcap r \equiv 0) \diamond \bigvee y \bigvee z\, \mathfrak{A}[y,z] \Rightarrow \mathfrak{F}[s,t,r]}{\Gamma, \langle\langle s,t\rangle, r\rangle \in mult \Rightarrow \mathfrak{F}[s,t,r]}
}
$$

QED

PROPOSITION 2.15. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.15i) $\langle\langle s^{\prime}, n\rangle, t\rangle \in add \Rightarrow \langle\langle s, n^{\prime}\rangle, t\rangle \in add$

(2.15ii) $\langle\langle c^{\prime}, a\rangle, n^{\prime}\rangle \in add \Rightarrow a \equiv 0 \diamond \ldots \diamond a \equiv n$

Proof. *Re* 2.15i. Employ an induction on n. As regards the induction basis, employ 2.10iii:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\langle\langle s^{\prime}, 0\rangle, t\rangle \in add \Rightarrow t \equiv s^{\prime}}{\langle\langle s^{\prime}, 0\rangle, t\rangle \in add \Rightarrow 0^{\prime} \equiv 0^{\prime} \sqcap t \equiv s^{\prime} \sqcap \langle\langle s, 0\rangle, s\rangle \in add}}{\langle\langle s^{\prime}, 0\rangle, t\rangle \in add \Rightarrow \bigvee y \bigvee z\, (0^{\prime} \equiv y^{\prime} \sqcap t \equiv z^{\prime} \sqcap \langle\langle s, y\rangle, z\rangle \in add)}}{\langle\langle s^{\prime}, 0\rangle, t\rangle \in add \Rightarrow (0^{\prime} \equiv 0 \sqcap t \equiv s) \diamond \bigvee y \bigvee z\, (0^{\prime} \equiv y^{\prime} \sqcap t \equiv z^{\prime} \sqcap \langle\langle s, y\rangle, z\rangle \in add)}}{\langle\langle s^{\prime}, 0\rangle, t\rangle \in add \Rightarrow \langle\langle s, 0^{\prime}\rangle, t\rangle \in add}
$$

As regards the induction step, firstly, employ 2.3ii:

$$\begin{array}{c} 0 \equiv n' \Rightarrow \\ \hline 0 \equiv n', t \equiv s' \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\ \hline\hline 0 \equiv n' \Box t \equiv s' \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \end{array}$$

Secondly, employ the induction hypothesis. In the proof figure to follow, let $\mathcal{C}$ stand for $(n'' \equiv 0 \Box t \equiv s)$:

$$\dfrac{t \equiv c' \Rightarrow t \equiv c'}{t \equiv c' \Box \langle\langle s', n\rangle, c\rangle \in add \Rightarrow t \equiv c'} \qquad \dfrac{\langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n'\rangle, c\rangle \in add}{t \equiv c' \Box \langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n'\rangle, c\rangle \in add}$$

$$\begin{array}{c} \hline t \equiv c' \Box \langle\langle s', n\rangle, c\rangle \in add \Rightarrow t \equiv c' \Box \langle\langle s, n'\rangle, c\rangle \in add \\ \hline t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow n'' \equiv n'' \Box t \equiv c' \Box \langle\langle s, n'\rangle, c\rangle \in add \\ \hline\hline t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \bigvee y \bigvee z\,(n'' \equiv y' \Box t \equiv z' \Box \langle\langle s, y\rangle, z\rangle \in add) \\ \hline t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \mathcal{C} \diamond \bigvee y \bigvee z\,(n'' \equiv y' \Box t \equiv z' \Box \langle\langle s, y\rangle, z\rangle \in add) \\ \hline t \equiv c' \Box \langle\langle s', n\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\ \hline n \equiv b, t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\ \hline n' \equiv b', t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\ \hline n' \equiv b' \Box t \equiv c' \Box \langle\langle s', b\rangle, c\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \\ \hline\hline \bigvee y \bigvee z\,(n' \equiv y' \Box t \equiv z' \Box \langle\langle s', y\rangle, z\rangle \in add) \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add \end{array}$$

Together:

$$\dfrac{(0 \equiv n' \Box t \equiv s') \diamond \bigvee y \bigvee z\,(n' \equiv y' \Box t \equiv z' \Box \langle\langle s', y\rangle, z\rangle \in add) \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add}{\langle\langle s', n'\rangle, t\rangle \in add \Rightarrow \langle\langle s, n''\rangle, t\rangle \in add}$$

Re 2.15ii. Employ an induction on n. I only consider the induction step. Let $\mathfrak{C}$ stand for $*_1 \equiv *_2 \diamond \cdots \diamond *_1 \equiv n'$:

$$\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
0 \equiv c' \Rightarrow \\ \hline
s \equiv 0 \,\square\, n' \equiv c' \Rightarrow s \equiv 0
\end{array}
&
\begin{array}{c}
\langle\langle c', a\rangle, n'\rangle \in add \Rightarrow a \equiv 0 \diamond \ldots \diamond a \equiv n \\ \hline\hline
\langle\langle c', a\rangle, n'\rangle \in add \Rightarrow a' \equiv 0' \diamond \ldots \diamond a' \equiv n' \\ \hline\hline
s \equiv a', n' \equiv b, \langle\langle c', a\rangle, b\rangle \in add \Rightarrow \mathfrak{E}[s, 0'] \\ \hline
s \equiv a', n'' \equiv b', \langle\langle c', a\rangle, b\rangle \in add \Rightarrow \mathfrak{E}[s, 0'] \\ \hline\hline
s \equiv a' \,\square\, n'' \equiv b' \,\square\, \langle\langle c', a\rangle, b\rangle \in add \Rightarrow \mathfrak{E}[s, 0'] \\ \hline\hline
\bigvee y \bigvee z\, (s \equiv y' \,\square\, n'' \equiv z' \,\square\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \mathfrak{E}[s, 0']
\end{array}
\; 2.3\mathrm{vii}
\end{array} \\ \hline
(s \equiv 0 \,\square\, n'' \equiv c') \diamond \bigvee y \bigvee z\, (s \equiv y' \,\square\, n'' \equiv z' \,\square\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \mathfrak{E}[s, 0] \\ \hline
\langle\langle c', s\rangle, n''\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n'
\end{array}$$

QED

The essential point for a representation of the least number operator is the availability of a smaller relation $<$ satisfying the following three conditions for all terms s and every numeral n:[8]

1. $\neg(s < 0)$
2. $s < n' \leftrightarrow s = 0 \vee \ldots \vee s = n$
3. $s < n \vee s = n \vee n < s$

i.e., a certain trichotomy of the natural numbers: two natural numbers are either equal or one of them is smaller than the other. In order to suit the present framework, the various notions involved have to be adapted. Equality will be replaced by identity, the inclusive successor will be replaced by the exclusive successor, $\vee$ will be replaced by $\diamond$. The task left is to find an appropriate notion of $<$. That's where a rudimentary[9] notion of natural number comes into play: m is smaller than n, if there is a natural number p such that $\mathrm{m} + \mathrm{p} = \mathrm{n}$.

What has to be accommodated for is a certain self-reference in the definition of the natural numbers which is expressed in the simple statement: n is a *natural number*, if it is either 0 or the successor of a natural number. In other words, *natural number* is defined in terms of itself. This is what the fixed point of the next proposition aims at.

PROPOSITION 2.16. *There is a term* $\mathbf{B}^*$ *satisfying*

$$\mathbf{L}^{\mathbf{i}}\mathbf{D}_\lambda \vdash \mathbf{B}^* = \lambda x\, (x \equiv 0 \diamond \bigvee y\, (y \in \mathbf{B}^* \,\square\, x \equiv y')) \,.$$

[8] Cf. [**13**], p. 40, proposition I.3.3. Note, however, that the $\wedge$ in the third condition listed there is obviously a typographical error that has to be replaced by $\vee$.

[9] 'Rudimentary', because full induction is not required.

Proof. This is an immediate consequence of the fixed point property as stated, *e.g.*, in [**14**], theorem 7.3, p. 382, or theorem 130.8 on p. 1779 of [**15**]. QED

COROLLARY 2.17. *Inferences according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-derivable*

$$\text{(2.17i)} \qquad \frac{s\in\lambda x(x\equiv 0\diamond\bigvee y(y\in\mathbf{B}^{*}\,\square\, x\equiv y^{\prime})),\varGamma\Rightarrow C}{s\in\mathbf{B}^{*},\varGamma\Rightarrow C}$$

$$\text{(2.17ii)} \qquad \frac{\varGamma\Rightarrow s\in\lambda x(x\equiv 0\diamond\bigvee y(y\in\mathbf{B}^{*}\,\square\, x\equiv y^{\prime}))}{\varGamma\Rightarrow s\in\mathbf{B}^{*}}$$

DEFINITION 2.18. $\bigvee^{\mathbf{B}^{*}}x\,\mathfrak{F}[x]:\equiv\bigvee x(x\in\mathbf{B}^{*}\,\square\,\mathfrak{F}[x])$.

I begin by listing the relevant properties of $\mathbf{B}^{*}$.

PROPOSITION 2.19. *Sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.19i) $\Rightarrow 0\in\mathbf{B}^{*}$

(2.19ii) $s\in\mathbf{B}^{*}\Rightarrow s^{\prime}\in\mathbf{B}^{*}$

Proof. Re 2.19i. Employ 2.16:

$$\cfrac{\cfrac{\cfrac{\Rightarrow 0\equiv 0}{\Rightarrow 0\equiv 0\diamond\bigvee^{\mathbf{B}^{*}}y\,(0\equiv y^{\prime})}}{\Rightarrow 0\in\lambda x\,(x\equiv 0\diamond\bigvee^{\mathbf{B}^{*}}y\,(x\equiv y^{\prime}))}}{\Rightarrow 0\in\mathbf{B}^{*}}\;2.17\text{ii}\,.$$

Re 2.19ii. Employ 2.16:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{s\in\mathbf{B}^{*}\Rightarrow s\in\mathbf{B}^{*}\qquad\Rightarrow s^{\prime}\equiv s^{\prime}}{s\in\mathbf{B}^{*}\Rightarrow s\in\mathbf{B}^{*}\,\square\, s^{\prime}\equiv s^{\prime}}}{s\in\mathbf{B}^{*}\Rightarrow\bigvee^{\mathbf{B}^{*}}y\,(s^{\prime}\equiv y^{\prime})}}{s\in\mathbf{B}^{*}\Rightarrow s^{\prime}\equiv 0\diamond\bigvee^{\mathbf{B}^{*}}y\,(s^{\prime}\equiv y^{\prime})}}{s\in\mathbf{B}^{*}\Rightarrow s^{\prime}\in\lambda x\,(x\equiv 0\diamond\bigvee^{\mathbf{B}^{*}}y\,(x\equiv y^{\prime}))}}{s\in\mathbf{B}^{*}\Rightarrow s^{\prime}\in\mathbf{B}^{*}}\;2.17\text{ii}\,.$$

QED

Proposition 2.20. *Inferences according to the following schema are* $\mathbf{L^{i}D}_{\lambda}$*-derivable.*

$$\frac{\Gamma \Rightarrow \mathfrak{F}[0] \qquad \Gamma, a\in\mathbf{B}^{*} \Rightarrow \mathfrak{F}[a^{\prime}]}{\Gamma, s\in\mathbf{B}^{*} \Rightarrow \mathfrak{F}[s]}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \mathfrak{F}[0]}{\Gamma, s\equiv 0 \Rightarrow \mathfrak{F}[s]} \qquad \dfrac{\dfrac{\dfrac{\Gamma, a\in\mathbf{B}^{*} \Rightarrow \mathfrak{F}[a^{\prime}]}{\Gamma, a\in\mathbf{B}^{*}, s\equiv a^{\prime} \Rightarrow \mathfrak{F}[s]}}{\Gamma, a\in\mathbf{B}^{*} \sqcap s\equiv a^{\prime} \Rightarrow \mathfrak{F}[s]}}{\Gamma, \bigvee^{\mathbf{B}^{*}} y\,(s\equiv y^{\prime}) \Rightarrow \mathfrak{F}[s]}}{\Gamma, s\equiv 0 \diamond \bigvee^{\mathbf{B}^{*}} y\,(s\equiv y^{\prime}) \Rightarrow \mathfrak{F}[s]}}{\Gamma, s\in\lambda x(x\equiv 0 \diamond \bigvee^{\mathbf{B}^{*}} y\,(x\equiv y^{\prime})) \Rightarrow \mathfrak{F}[s]}}{\Gamma, s\in\mathbf{B}^{*} \Rightarrow \mathfrak{F}[s]}\;2.17\text{i}\,.$$

QED

For minimization a smaller-relation between numerals is required which is introduced next (essentially taken from [**18**], p. 8):

Definition 2.21. $less :\equiv \lambda xy \bigvee^{\mathbf{B}^{*}} z\,(\langle\langle z^{\prime}, x\rangle, y\rangle \in add)\,.$

Proposition 2.22. *If* m *and* n *are two natural numbers such that* m < n*, then* $\Rightarrow \langle m, n\rangle \in less$ *is* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

Proof. If m < n, then there is a natural number p such that p′ + m = n. By 2.19, $\Rightarrow p\in\mathbf{B}^{*}$ is $\mathbf{L^{i}D}_{\lambda}$-deducible and by the numeralwise representability of addition, $\Rightarrow \langle\langle p^{\prime}, m\rangle, n\rangle \in add$ is $\mathbf{L^{i}D}_{\lambda}$-deducible.

$$\dfrac{\dfrac{\dfrac{\Rightarrow p\in\mathbf{B}^{*} \qquad \Rightarrow \langle\langle p^{\prime}, m\rangle, n\rangle \in add}{\Rightarrow p\in\mathbf{B}^{*} \sqcap \langle\langle p^{\prime}, m\rangle, n\rangle \in add}}{\Rightarrow \bigvee^{\mathbf{B}^{*}} z\,(\langle\langle z^{\prime}, m\rangle, n\rangle \in add)}}{\Rightarrow \langle m, n\rangle \in \lambda xy \bigvee^{\mathbf{B}^{*}} z\,(\langle\langle z^{\prime}, x\rangle, y\rangle \in add)}$$

QED

Proposition 2.23. *If* n *is a natural number, then sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.23i) $t\in\mathbf{B}^{*} \Rightarrow \langle 0, t^{\prime}\rangle \in less$

(2.23ii) $s\in\mathbf{B}^{*}, \langle\langle s^{\prime\prime}, n\rangle, t\rangle \in add \Rightarrow \langle n^{\prime}, t\rangle \in less$

(2.23iii) $\langle n, s\rangle \in less \Rightarrow n' \equiv s \diamond \langle n', s\rangle \in less$

(2.23iv) $n \equiv s \Rightarrow \langle s, n'\rangle \in less$

(2.23v) $\langle s, 0\rangle \in less \Rightarrow$

(2.23vi) $\langle s, n'\rangle \in less \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n$

(2.23vii) $\langle s, n\rangle \in less \Rightarrow \langle s, n'\rangle \in less$

Proof. *Re* 2.23i. Employ 2.10i:

$$\dfrac{\dfrac{\dfrac{t \in \mathbf{B}^* \Rightarrow t \in \mathbf{B}^* \qquad \Rightarrow \langle\langle t', 0\rangle, t'\rangle \in add}{t \in \mathbf{B}^* \Rightarrow t \in \mathbf{B}^* \,\square\, \langle\langle t', 0\rangle, t'\rangle \in add}}{t \in \mathbf{B}^* \Rightarrow \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', 0\rangle, t'\rangle \in add)}}{t \in \mathbf{B}^* \Rightarrow \langle 0, t'\rangle \in less}$$

Re 2.23ii. Employ 2.15i:

$$\dfrac{\dfrac{s \in \mathbf{B}^*, \langle\langle s'', n\rangle, t\rangle \in add \Rightarrow s \in \mathbf{B}^* \,\square\, \langle\langle s', n'\rangle, t\rangle \in add}{s \in \mathbf{B}^*, \langle\langle s'', n\rangle, t\rangle \in add \Rightarrow \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', n'\rangle, t\rangle \in add)}}{s \in \mathbf{B}^*, \langle\langle s'', n\rangle, t\rangle \in add \Rightarrow \langle n', t\rangle \in less}$$

Re 2.23iii. Employ an induction on n. As regards the induction basis, employ 2.10iii and 2.23ii. Let $\mathcal{C}$ stand for $0' \equiv s \diamond \langle 0', s\rangle \in less$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\langle\langle 0', 0\rangle, s\rangle \in add \Rightarrow 0' \equiv s}{\langle\langle 0', 0\rangle, s\rangle \in add \Rightarrow \mathcal{C}} \qquad \dfrac{a \in \mathbf{B}^*, \langle\langle a'', 0\rangle, s\rangle \in add \Rightarrow \langle 0', s\rangle \in less}{a \in \mathbf{B}^*, \langle\langle a'', 0\rangle, s\rangle \in add \Rightarrow \mathcal{C}}}{c \in \mathbf{B}^*, \langle\langle c', 0\rangle, s\rangle \in add \Rightarrow 0' \equiv s \diamond \langle 0', s\rangle \in less}}{c \in \mathbf{B}^* \,\square\, \langle\langle c', 0\rangle, s\rangle \in add \Rightarrow 0' \equiv s \diamond \langle 0', s\rangle \in less}}{\bigvee^{\mathbf{B}^*} z\,(\langle\langle z', 0\rangle, s\rangle \in add) \Rightarrow 0' \equiv s \diamond \langle 0', s\rangle \in less}}{\langle 0, s\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow 0' \equiv s \diamond \langle 0', s\rangle \in less}$$

As regards the induction step, employ again 2.10iii and 2.23ii. Let $\mathcal{C}$ stand for $n'' \equiv s \diamond \langle n'', s\rangle \in less$:

$$
\dfrac{\dfrac{\dfrac{\langle\langle 0', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s}{\langle\langle 0', n'\rangle, s\rangle \in add \Rightarrow \mathcal{C}} \qquad \dfrac{a \in \mathbf{B}^*, \langle\langle a'', n'\rangle, s\rangle \in add \Rightarrow \langle n'', s\rangle \in less}{a \in \mathbf{B}^*, \langle\langle a'', n'\rangle, s\rangle \in add \Rightarrow \mathcal{C}}}{\dfrac{c \in \mathbf{B}^*, \langle\langle c', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less}{\dfrac{c \in \mathbf{B}^* \,\square\, \langle\langle c', n'\rangle, s\rangle \in add \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less}{\bigvee^{\mathbf{B}^*} z_1 \,(\langle\langle {z_1}', n'\rangle, s\rangle \in add) \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less}}}}{\langle n', s\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow n'' \equiv s \diamond \langle n'', s\rangle \in less}
$$

Re 2.23v.

$$
\begin{array}{c}
\begin{array}{c} 0 \equiv c' \Rightarrow \\ \hline s \equiv 0 \wedge 0 \equiv c' \Rightarrow \end{array}
\qquad
\begin{array}{c} 0 \equiv b' \Rightarrow \\ \hline\hline s \equiv a' \wedge 0 \equiv b' \wedge \langle\langle c', a\rangle, b\rangle \in add \Rightarrow \\ \hline\hline \bigvee y \bigvee z\,(s \equiv y' \,\square\, 0 \equiv z' \,\square\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \end{array} \\
\hline
(s \equiv 0 \,\square\, 0 \equiv c') \diamond \bigvee y \bigvee z\,(s \equiv y' \,\square\, 0 \equiv z' \,\square\, \langle\langle c', y\rangle, z\rangle \in add) \Rightarrow \\
\hline
\langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\
\hline
c \in \mathbf{B}^*, \langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\
\hline
c \in \mathbf{B}^* \,\square\, \langle\langle c', s\rangle, 0\rangle \in add \Rightarrow \\
\hline
\bigvee^{\mathbf{B}^*} z\,(\langle\langle z', s\rangle, 0\rangle \in add) \Rightarrow \\
\hline
\langle s, 0\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow
\end{array}
$$

Re 2.23vi. Employ 2.15ii:

$$
\begin{array}{c}
\langle\langle c', s\rangle, n'\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\
\hline
c \in \mathbf{B}^*, \langle\langle c', s\rangle, n\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\
\hline
c \in \mathbf{B}^* \,\square\, \langle\langle c', s\rangle, n\rangle \in add \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\
\hline
\bigvee^{\mathbf{B}^*} z\,(\langle\langle z', s\rangle, n'\rangle \in add) \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n \\
\hline
\langle s, n'\rangle \in \lambda xy \bigvee^{\mathbf{B}^*} z\,(\langle\langle z', x\rangle, y\rangle \in add) \Rightarrow s \equiv 0 \diamond \ldots \diamond s \equiv n
\end{array}
$$

Re 2.23vii. Distinguish two cases according to whether $\mathrm{n} = 0$ or $\mathrm{n} = \mathrm{p}'$, $\mathrm{p} \in \mathbb{N}$. The first case is an immediate consequence of 2.23v. As regards the second case, employ 2.23vi and 2.22:

$$\dfrac{\langle s,p'\rangle \in less \Rightarrow s\equiv 0 \diamond \cdots \diamond s \equiv p \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \langle 0,p''\rangle \in less}{s\equiv 0 \Rightarrow \langle s,p''\rangle \in less} \quad \dfrac{\Rightarrow \langle 0',p''\rangle \in less}{s\equiv 0' \Rightarrow \langle s,p''\rangle \in less}}{s\equiv 0 \diamond s\equiv 0' \Rightarrow \langle s,p''\rangle \in less}}{\text{p analogous } \diamond\text{-introductions}}}{s\equiv 0 \diamond \cdots \diamond s\equiv p \Rightarrow \langle s,p''\rangle \in less}}{\langle s,p'\rangle \in less \Rightarrow \langle s,p''\rangle \in less} \clubsuit$$

QED

PROPOSITION 2.24. *For all natural numbers* n, *sequents according to the following schemata are* $\mathbf{L^iD}_\lambda$*-deducible.*

(2.24i) $s\in\mathbf{B}^* \Rightarrow \langle s,0\rangle \in less \diamond s\equiv 0 \diamond \langle 0,s\rangle \in less$

(2.24ii) $\langle s,n\rangle \in less \diamond s\equiv n \diamond \langle n,s\rangle \in less \Rightarrow$
$\langle s,n'\rangle \in less \diamond s\equiv n' \diamond \langle n',s\rangle \in less$

(2.24iii) $s\in\mathbf{B}^* \Rightarrow \langle s,n\rangle \in less \diamond s\equiv n \diamond \langle n,s\rangle \in less$

Proof. Re 2.24i. Employ 2.23i:

$$\dfrac{\dfrac{\Rightarrow 0\equiv 0}{\Rightarrow \langle 0,0\rangle \in less \diamond 0\equiv 0 \diamond \langle 0,0\rangle \in less} \quad \dfrac{a\in\mathbf{B}^* \Rightarrow \langle 0,a'\rangle \in less}{a\in\mathbf{B}^* \Rightarrow \langle a',0\rangle \in less \diamond a'\equiv 0 \diamond \langle 0,a'\rangle \in less}}{s\in\mathbf{B}^* \Rightarrow \langle s,0\rangle \in less \diamond s\equiv 0 \diamond \langle 0,s\rangle \in less}.$$

Re 2.24ii. By 2.23vii

$$\dfrac{\langle s,n\rangle \in less \Rightarrow \langle s,n'\rangle \in less}{\langle s,n\rangle \in less \Rightarrow \langle s,n'\rangle \in less \diamond s = n' \diamond \langle n',s\rangle \in less},$$

and by 2.23iv

$$\dfrac{s\equiv n \Rightarrow \langle s,n'\rangle \in less}{\langle s,n\rangle \in less \Rightarrow \langle s,n'\rangle \in less \diamond s = n' \diamond \langle n',s\rangle \in less}.$$

and by 2.23iii

$$\dfrac{\langle n,s\rangle \in less \Rightarrow n'\equiv s \diamond \langle n',s\rangle \in less}{\langle s,n\rangle \in less \Rightarrow \langle s,n'\rangle \in less \diamond s = n' \diamond \langle n',s\rangle \in less}.$$

Together:

$$\langle s,n\rangle \in less \diamond s\equiv n \diamond \langle n,s\rangle \in less \Rightarrow \langle s,n'\rangle \in less \diamond s\equiv n' \diamond \langle n',s\rangle \in less$$

Re 2.24iii. Employing 2.24i and 2.24ii for a metatheoretical induction gives

$$s\in\mathbf{B}^{*}\Rightarrow\langle s,n\rangle\in \mathit{less}\diamond s\equiv n\diamond\langle n,s\rangle\in \mathit{less}$$

for all natural numbers n. QED

COROLLARY 2.25. *Inferences according to the following schema are* $\mathbf{L^{i}D}_{\lambda}$*-derivable*

$$\dfrac{\langle s,n\rangle\in \mathit{less}\diamond s\equiv n\diamond\langle n,s\rangle\in \mathit{less},\Gamma\Rightarrow C}{s\in\mathbf{B}^{*},\Gamma\Rightarrow C}$$

PROPOSITION 2.26. *If* m *is a natural number,* $\vec{n}$ *a* p*-tuple of natural numbers and* $\mathfrak{C}:\equiv\langle\langle *_1,*_2\rangle,0\rangle\in s\,\square\bigwedge z\,(\langle z,*_2\rangle\in \mathit{less}\rightarrow\langle\langle *_1,z\rangle,0\rangle\notin s)$*, then sequents according to the following schemata are* $\mathbf{L^{i}D}_{\lambda}$*-deducible.*

(2.26i) $\langle\langle\vec{n},m\rangle,0\rangle\in s,m\equiv 0\Rightarrow\mathfrak{C}[\vec{n},m]$

(2.26ii) $\langle\langle\vec{n},c\rangle,0\rangle\in s,\langle c,k'\rangle\in \mathit{less}\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s$

(2.26iii) $\langle\langle\vec{n},m\rangle,0\rangle\in s,\langle m,c\rangle\in \mathit{less},\mathfrak{C}[\vec{n},m]\Rightarrow$

Proof. Re 2.26i. Employ 2.23v:

$$\dfrac{\langle\langle\vec{n},m\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},m\rangle,0\rangle\in s\qquad\dfrac{\dfrac{\dfrac{\dfrac{\langle a,0\rangle\in \mathit{less}\Rightarrow}{m\equiv 0,\langle a,m\rangle\in \mathit{less}\Rightarrow}}{m\equiv 0,\langle a,m\rangle\in \mathit{less}\Rightarrow\langle\langle\vec{n},a\rangle,0\rangle\notin s)}}{m\equiv 0\Rightarrow\langle a,m\rangle\in \mathit{less}\rightarrow\langle\langle\vec{n},a\rangle,0\rangle\notin s)}}{m\equiv 0\Rightarrow\bigwedge z_1\,(\langle z_1,m\rangle\in \mathit{less}\rightarrow\langle\langle\vec{n},z_1\rangle,0\rangle\notin s)}}{\langle\langle\vec{n},m\rangle,0\rangle\in s,m\equiv 0\Rightarrow\langle\langle\vec{n},m\rangle,0\rangle\in s\,\square\bigwedge z_1\,(\langle z_1,m\rangle\in \mathit{less}\rightarrow\langle\langle\vec{n},z_1\rangle,0\rangle\notin s)}\,.$$

Re 2.26ii. Let $\mathcal{A}$ stand for $c\equiv 0\diamond\ldots\diamond c\equiv k$, $k\geq 1$. Employ 2.23vi:

$$\dfrac{\langle c,k'\rangle\in \mathit{less}\Rightarrow\mathcal{A}\qquad\dfrac{\dfrac{c\equiv 0,\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\qquad c\equiv 0',\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0'\rangle,0\rangle\in s}{\dfrac{c\equiv 0\diamond c\equiv 0',\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}{\text{k}-1\text{ analogous }\diamond\text{-introductions left}}}}{\mathcal{A},\langle\langle\vec{n},c\rangle,0\rangle\in s\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}}{\langle\langle\vec{n},c\rangle,0\rangle\in s,\langle c,k'\rangle\in \mathit{less}\Rightarrow\langle\langle\vec{n},0\rangle,0\rangle\in s\diamond\cdots\diamond\langle\langle\vec{n},k\rangle,0\rangle\in s}\,\clubsuit.$$

Re 2.26iii.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\langle m,c\rangle\in \mathit{less} \Rightarrow \langle m,c\rangle\in \mathit{less} \qquad \dfrac{\langle\langle\vec{n}\,,m\rangle,0\rangle\in s \Rightarrow \langle\langle\vec{n}\,,m\rangle,0\rangle\in s}{\langle\langle\vec{n}\,,m\rangle,0\rangle\in s, \langle\langle\vec{n}\,,m\rangle,0\rangle\notin s \Rightarrow}}{\langle\langle\vec{n}\,,m\rangle,0\rangle\in s, \langle m,c\rangle\in \mathit{less}, \langle m,c\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n}\,,m\rangle,0\rangle\notin s \Rightarrow}}{\langle\langle\vec{n}\,,m\rangle,0\rangle\in s, \langle m,c\rangle\in \mathit{less}, \bigwedge z\,(\langle z,c\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n}\,,z\rangle,0\rangle\notin s) \Rightarrow}}{\langle\langle\vec{n},m\rangle,0\rangle\in s, \langle m,c\rangle\in \mathit{less}, \langle\langle\vec{n},m\rangle,0\rangle\in s, \bigwedge z\,(\langle z,c\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n},z\rangle,0\rangle\notin s) \Rightarrow}}{\langle\langle\vec{n}\,,m\rangle,0\rangle\in s, \langle m,c\rangle\in \mathit{less}, \mathfrak{C}[\vec{n}\,,m] \Rightarrow}
$$

QED

Definition 2.27.
$\mathit{min}[s] :\equiv \lambda xy\,(y\in\mathbf{B}^* \,\square\, \langle\langle\vec{x},y\rangle,0\rangle\in s \,\square\, \bigwedge z\,(\langle z,y\rangle\in \mathit{less} \rightarrow \langle\langle\vec{x},z\rangle,0\rangle\notin s))\,.$

Proposition 2.28. *If the function* $\mathfrak{g}(\vec{\mathrm{x}},\mathrm{y})$ *is numeralwise represented in* $\mathbf{L^iD}_\lambda$ *by the term* g*, and the function* $\mathfrak{f}(\vec{\mathrm{x}}) = \mu\mathrm{y}\,(\mathfrak{g}(\vec{\mathrm{x}},\mathrm{y})=0)$ *obtained from* $\mathfrak{g}$ *by* μ*-recursion is total, then* $\mathfrak{f}$ *is numeralwise represented in* $\mathbf{L^iD}_\lambda$ *by* $\mathit{min}[g]$*.*

Proof. If $\mathfrak{g}(\vec{\mathrm{n}},\mathrm{m}) = 0$ and $\mathfrak{f}(\vec{\mathrm{n}}) = \mathrm{m}$, *i.e.*, $\mu\mathrm{y}\,(\mathrm{g}(\vec{\mathrm{n}},\mathrm{y}) = 0) = \mathrm{m}$, then by the assumption that $\mathfrak{g}$ is numeralwise represented in $\mathbf{L^iD}_\lambda$ by g, we have that

(2.28i) $\Rightarrow \langle\langle\vec{n}\,,m\rangle,0\rangle\in g,$ and

(2.28ii) $\Rightarrow \bigwedge x\,(\langle\langle\vec{n}\,,m\rangle,x\rangle\in g \rightarrow x\equiv 0)$

(2.28iii) $\langle\langle\vec{n}\,,i\rangle,0\rangle\in g \Rightarrow$ if for all $\mathrm{i} < \mathrm{m}$

are $\mathbf{L^iD}_\lambda$-deducible. In addition, 2.28iii yields:

(2.28iv) $\langle\langle\vec{n}\,,0\rangle,0\rangle\in g \diamond \cdots \diamond \langle\langle\vec{n}\,,k\rangle,0\rangle\in g \Rightarrow$

for $\mathrm{k}' = \mathrm{m}$ by successive $\diamond$-introduction. (There is no $\mathrm{i} < \mathrm{m}$ for $\mathrm{m} = 0$.)

Now, what has to be shown for the numeralwise representability of minimization is that

$$\Rightarrow \langle\vec{n}\,,m\rangle\in \mathit{min}[g], \quad \text{and}$$
$$\Rightarrow \bigwedge x\,(\langle\vec{n}\,,x\rangle\in \mathit{min}[g] \rightarrow x\equiv m)\,.$$

are $\mathbf{L^iD}_\lambda$-deducible. First of all, cutting 2.28i with 2.26i yields

$$m\equiv 0 \Rightarrow \bigwedge z_1\,(\langle z_1,m\rangle\in \mathit{less} \rightarrow \langle\langle\vec{n}\,,z_1\rangle,0\rangle\notin s)\,, \quad \text{and}$$

cutting 2.26ii and 2.28iv gives way to the following deduction:

$$\begin{array}{c}
\langle\langle\vec{n}\,,c\rangle,0\rangle\in g,\langle c,k'\rangle\in less\Rightarrow \\ \hline
\langle c,k'\rangle\in less\Rightarrow\langle\langle\vec{n}\,,c\rangle,0\rangle\notin g \\ \hline
\langle c,m\rangle\in less, m\equiv k'\Rightarrow\langle\langle\vec{n}\,,c\rangle,0\rangle\notin g \\ \hline
m\equiv k'\Rightarrow\langle c,m\rangle\in less\rightarrow\langle\langle\vec{n}\,,c\rangle,0\rangle\notin g \\ \hline
m\equiv k'\Rightarrow\bigwedge z\,(\langle z,m\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin g)
\end{array}$$

This yields

$m\equiv 0\Rightarrow\langle\vec{n}\,,m\rangle\in min[g]\,,$ and

$m\equiv k'\Rightarrow\langle\vec{n}\,,m\rangle\in min[g]$

in the following way (where Γ is $m\equiv 0$, $m\equiv k'$, resp.), employing 2.28i:

$$\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{c}\Rightarrow\langle\langle\vec{n}\,,m\rangle,0\rangle\in g \qquad \Gamma\Rightarrow\bigwedge z\,(\langle z,m\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin g) \\ \hline
\Gamma\Rightarrow\langle\langle\vec{n}\,,m\rangle,0\rangle\in g\sqcap\bigwedge z\,(\langle z,m\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin g)\end{array} \\
\Rightarrow m\in\mathbf{B}^* &
\end{array} \\ \hline\hline
\Gamma\Rightarrow m\in\mathbf{B}^*\sqcap\langle\langle\vec{n}\,,m\rangle,0\rangle\in g\sqcap\bigwedge z\,(\langle z,m\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin g)) \\ \hline\hline
\Gamma\Rightarrow\langle\vec{n}\,,m\rangle\in\lambda\vec{x}\,y\,(y\in\mathbf{B}^*\sqcap\langle\langle\vec{x}\,,y\rangle,0\rangle\in g\sqcap\bigwedge z\,(\langle z,y\rangle\in less\rightarrow\langle\langle\vec{x}\,,z\rangle,0\rangle\notin g))
\end{array}$$

Cutting 2.26ii and 2.28iv gives way to the following deduction, where $\mathfrak{C}$ is the nominal form $\langle\langle\vec{n}\,,*_1\rangle,0\rangle\in g$:

$$\begin{array}{c}
\langle\langle\vec{n}\,,c\rangle,0\rangle\in g,\langle c,k'\rangle\in less\Rightarrow\mathfrak{C}[0]\diamond\cdots\diamond\mathfrak{C}[k] \qquad \mathfrak{C}[0]\diamond\cdots\diamond\mathfrak{C}[k]\Rightarrow \\ \hline\hline
\langle\langle\vec{n}\,,c\rangle,0\rangle\in g,\langle c,k'\rangle\in less\Rightarrow \quad \clubsuit \\ \hline\hline
m\in\mathbf{B}^*,\langle\langle\vec{n}\,,c\rangle,0\rangle\in g,\bigwedge z\,(\langle z,c\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin s),\langle c,k'\rangle\in less\Rightarrow \\ \hline\hline
m\in\mathbf{B}^*\sqcap\langle\langle\vec{n}\,,c\rangle,0\rangle\in g\sqcap\bigwedge z\,(\langle z,c\rangle\in less\rightarrow\langle\langle\vec{n}\,,z\rangle,0\rangle\notin s),\langle c,k'\rangle\in less\Rightarrow \\ \hline
\langle\vec{n}\,,c\rangle\in min[g],\langle c,k'\rangle\in less\Rightarrow
\end{array}$$

Since

$\langle\vec{n}\,,c\rangle\in min[g],\langle c,0\rangle\in less\Rightarrow$

holds almost trivially as a consequence of

$\langle c,0\rangle\in less\Rightarrow$ (2.23v) ,

and m is either 0 or k′ for some natural number k, we have that

$\langle\vec{n}\,,c\rangle\in min[g],\langle c,m\rangle\in less\Rightarrow$

is $\mathbf{L^iD}_\lambda$-deducible.

The last preparational step is to provide

$$\langle\vec{n}, c\rangle \in min[g], \langle m, c\rangle \in less \Rightarrow \ .$$

which is readily obtained from 2.28ii and 2.26iii by means of a cut.

This now yields the second condition of numeralwise representability of minimization in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{c \equiv m \Rightarrow c \equiv m \qquad \dfrac{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \Rightarrow \qquad \langle\vec{n}, c\rangle \in min[g], \langle m, c\rangle \in less \Rightarrow}{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \diamond \langle m, c\rangle \in less \Rightarrow}}{\langle\vec{n}, c\rangle \in min[g], \langle c, m\rangle \in less \diamond \langle m, c\rangle \in less \diamond c \equiv m \Rightarrow c \equiv m}}{\langle\vec{n}, c\rangle \in min[g] \Rightarrow c \equiv m}\,2.25}{\Rightarrow \langle\vec{n}, c\rangle \in min[g] \rightarrow c \equiv m}}{\Rightarrow \bigwedge x\, (\langle\vec{n}, x\rangle \in min[g] \rightarrow x \equiv m)}\ .$$

QED

THEOREM 2.29. *The recursive functions are numeralwise representable in* $\mathbf{L^iD}_\lambda$.

Proof. As for result 45.46 in [**15**], p. 573, this is an immediate consequence of the numeralwise representability of addition, multiplication, the identity functions, the characteristic function of equality, composition, and minimization. QED

THEOREM 2.30. $\mathbf{L^iD}_\lambda$ *is essentially undecidable.*

Proof. As for any consistent theory which allows numeralwise representability of all recursive functions.[10] QED

REMARK 2.31. In view of the cut eliminability in $\mathbf{L^iD}_\lambda$, the foregoing two results extend to $\mathbf{LP}_\lambda$.

3. Addition 130f. Fixed points and denotational devices

Definite description can only be established in a somewhat reduced form in $\mathbf{L^iD}_\lambda$, the reason being a contraction that sneaks into the proof of proposition 41.17 in [**15**], p. 470. Could this contraction possibly do harm? In the present section I shall show that it actually does.

[10] Cf. theorem 48.27 in [**15**], p. 612, for the paradigm of proof.

The failure of extensionality may already be seen as an indication that denotation is not quite as straightforward a business as was thought in the early days of modern logic. The following application of the fixed point property to establish the incompatibility of indefinite description (ε-operator) with $\mathbf{L^{i}D}_{\lambda}$ may well be seen as contributing to this view.

PROPOSITION 3.1. $\mathbf{L^{i}D}_{\lambda} \cup \{\bigvee x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[\varepsilon x\, \mathfrak{F}[x]]\} \vdash \bot$

Proof. Take the fixed point $\phi = \varepsilon x\,(\phi \neq x)$ and consider the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bigvee x\,(\phi \neq x) \Rightarrow \phi \neq \varepsilon x\,(\phi \neq x) \qquad \dfrac{\Rightarrow \phi = \varepsilon x\,(\phi \neq x)}{\phi \neq \varepsilon x\,(\phi \neq x) \Rightarrow}}{\bigvee x\,(\phi \neq x) \Rightarrow}\,\clubsuit}{\phi \neq 0 \Rightarrow}}{\phi = 1, 1 \neq 0 \Rightarrow}}{\phi = 1 \Rightarrow}}{\Rightarrow \phi \neq 1} \qquad \dfrac{\begin{matrix}\text{as on the left}\\ \vdots \\ \bigvee x\,(\phi \neq x) \Rightarrow\end{matrix}}{\phi \neq 1 \Rightarrow}}{\Rightarrow}\,\clubsuit$$

QED

REMARK 3.2. Notice that there is no contraction involved in this deduction. They are hiding in the ε-initial sequent.[11]

That the ε-operator is not compatible with $\mathbf{L^{i}D}_{\lambda}$ may not surprise people who find the ε-operator outrageous anyway; so I shall show that the least number operator doesn't fare any better.

First of all: the formulation of the least number operator has to be restricted to natural numbers. But in view of the fact that only 0 and 1 are employed in the proof above, this is little more than a formality. The definition of the natural numbers provided in 41.60 on p. 487 of [**15**] would actually do, since even without contraction it still yields 0 and 1 as natural numbers.

PROPOSITION 3.3. $\mathbf{L^{i}D}_{\lambda} \cup \{\bigvee^{\mathrm{N}} x\, \mathfrak{F}[x] \Rightarrow \mathfrak{F}[\mu x\, \mathfrak{F}[x]]\} \vdash \bot$.

[11] This has been used as a convenient way of "proving" that abandoning contraction is no safeguard against the paradoxes.

Proof. Take the fixed point $\phi = \mu x\,(\phi \neq x)$ and consider the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0\in\mathbf{N} \qquad \dfrac{\dfrac{\dfrac{\bigvee^{\mathrm{N}}x\,(\phi\neq x)\Rightarrow\phi\neq\mu x\,(\phi\neq x) \qquad \dfrac{\Rightarrow\phi=\mu x\,(\phi\neq x)}{\phi\neq\mu x\,(\phi\neq x)\Rightarrow}}{\bigvee^{\mathrm{N}}x\,(\phi\neq x)\Rightarrow}\clubsuit}{0\in\mathbf{N}\,\square\,\phi\neq 0\Rightarrow}}{0\in\mathbf{N},\phi\neq 0\Rightarrow}}{\phi\neq 0\Rightarrow}\clubsuit}{\phi=1,1\neq 0\Rightarrow}}{\phi=1\Rightarrow}}{\Rightarrow\phi\neq 1} \qquad \dfrac{\Rightarrow 1\in\mathbf{N} \qquad \dfrac{\dfrac{\begin{array}{c}\text{as on the left}\\ \vdots\\ \bigvee^{\mathrm{N}}x\,(\phi\neq x)\Rightarrow\end{array}}{1\in\mathbf{N}\,\square\,\phi\neq 1\Rightarrow}}{1\in\mathbf{N},\phi\neq 1\Rightarrow}\clubsuit}{\phi\neq 1\Rightarrow}}{\Rightarrow}\clubsuit$$

QED

Still, someone who objects to the least number operator on the basis of the non-finite character of the least number principle[12] might not find that too surprising. So I shall go one step further and show that even without the least number principle one obtains a contradiction.

PROPOSITION 3.4.

$$\mathbf{L^iD}_\lambda \cup \{\textstyle\bigvee^{\mathrm{N}}x\,(\mathfrak{F}[x]\,\square\,\bigwedge^{\mathrm{N}}y\,(y<x\to\neg\mathfrak{F}[y]))\Rightarrow \mathfrak{F}[\mu x\,\mathfrak{F}[x]]\}\vdash\bot\,.$$

Proof. Take again the fixed point $\phi = \mu x\,(\phi \neq x)$ and consider the following variation of the by now familiar deduction which yields $\bigvee^{\mathrm{N}}x\,(\phi \neq x\,\square\,\bigwedge^{\mathrm{N}}y\,(y < x \to \neg(\phi \neq y)))) \Rightarrow$ instead of $\bigvee^{\mathrm{N}}x\,(\phi \neq x) \Rightarrow$ and continue as follows:

$$\dfrac{\Rightarrow 0\in\mathbf{N} \qquad \dfrac{\dfrac{\dfrac{b\in\mathbf{N},b<0\Rightarrow\neg(\phi\neq b)}{b\in\mathbf{N}\Rightarrow b<0\to\neg(\phi\neq b)}}{\Rightarrow\bigwedge^{\mathrm{N}}y\,(y<0\to\neg(\phi\neq y))} \qquad \dfrac{\dfrac{\bigvee^{\mathrm N}x\,(\phi\neq x\,\square\,\bigwedge^{\mathrm N}y\,(y<x\to\neg(\phi\neq y))))\Rightarrow}{0\in\mathbf N\,\square\,\phi\neq 0\,\square\,\bigwedge^{\mathrm N}y\,(y<0\to\neg(\phi\neq y))))\Rightarrow}}{0\in\mathbf N,\phi\neq0,\bigwedge^{\mathrm N}y\,(y<0\to\neg(\phi\neq y))))\Rightarrow}}{0\in\mathbf N,\phi\neq 0\Rightarrow}\clubsuit}{\phi\neq 0\Rightarrow}\clubsuit$$

[12] Cf. proposition 46.20 in [**15**] for the least number principle in first order arithmetic. As regards the non-finite character of the least number principle, cf. Bernays in quotation 80.6 in the [**15**].

This result can now be employed to yield $\Rightarrow \phi \neq 1$ as usual, but also to prove $\bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y))))$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{b \in \mathbf{N}, b < 1 \Rightarrow b = 0 \qquad \dfrac{\phi \neq 0 \Rightarrow}{b = 0, \phi \neq b \Rightarrow}}{b \in \mathbf{N}, b < 1, \phi \neq b \Rightarrow}\clubsuit}{b \in \mathbf{N}, b < 1 \Rightarrow \phi \neq b}}{b \in \mathbf{N} \Rightarrow b < 1 \rightarrow \phi \neq b}}{\Rightarrow \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y))}$$

Continue as above, only with 1 instead of 0:

$$\dfrac{\Rightarrow 1 \in \mathbf{N} \qquad \dfrac{\Rightarrow \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)) \qquad \dfrac{\dfrac{\bigvee^{\mathrm{N}} x\,(\phi \neq x \,\square\, \bigwedge^{\mathrm{N}} y\,(y < x \rightarrow \neg(\phi \neq y))) \Rightarrow}{1 \in \mathbf{N} \,\square\, \phi \neq 1 \,\square\, \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)) \Rightarrow}}{1 \in \mathbf{N}, \phi \neq 1, \bigwedge^{\mathrm{N}} y\,(y < 1 \rightarrow \neg(\phi \neq y)))) \Rightarrow}}{1 \in \mathbf{N}, \phi \neq 1 \Rightarrow}\clubsuit}{\phi \neq 1 \Rightarrow}\clubsuit$$

QED

This result can be extended to the ι-operator with the usual initial sequent. The point is that the least number operator is just a special form of definite description and the least number principle and the only natural numbers actually employed are 0 and 1.

Convention 3.5.
$\mathfrak{C}[s] :\equiv s \in \{0,1\} \,\square\, \mathfrak{F}[s] \,\square\, \bigwedge y\,(y \in \{0,1\} \,\square\, y < s \rightarrow \neg\mathfrak{F}[y])\,.$

Proposition 3.6. *If $\mathfrak{C}[s]$ is according to convention 3.5, then sequents according to the following schemata are $\mathbf{L^iD}_\lambda$-deducible.*

(3.6i) $\quad a \equiv 0, b \equiv 0 \Rightarrow a = b$

(3.6ii) $\quad a \equiv 1, b \equiv 1 \Rightarrow a = b$

(3.6iii) $\quad a \equiv 0, b \equiv 1, \mathfrak{F}[a], \bigwedge y\,(y \in \{0,1\} \,\square\, y < b \rightarrow \neg\mathfrak{F}[y]) \Rightarrow a = b$

(3.6iv) $\quad a \equiv 1, b \equiv 0, \mathfrak{F}[b], \bigwedge y\,(y \in \{0,1\} \,\square\, y < a \rightarrow \neg\mathfrak{F}[y]) \Rightarrow a = b$

(3.6v) $\quad \mathfrak{C}[a], \mathfrak{C}[b] \Rightarrow a = b$

(3.6vi) $\quad \bigvee x\, \mathfrak{C}[x] \Rightarrow \bigvee x\,(\mathfrak{C}[x] \wedge \bigwedge y\,(\mathfrak{C}[y] \rightarrow x = y))$

(3.6vii) $\quad \bigvee x\, \mathfrak{C}[x] \Rightarrow \bigvee x\, \mathfrak{C}[x] \,\square\, \bigwedge x \bigwedge y\,(\mathfrak{C}[x] \,\square\, \mathfrak{C}[y] \rightarrow x = y)$

Proof. *Re* 3.6i and ii. Trivial.
Re 3.6iii

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0\in\{0,1\} \qquad \Rightarrow 0<1}{\Rightarrow 0\in\{0,1\}\,\Box\, 0<1}}{a\equiv 0, b\equiv 1 \Rightarrow a\in\{0,1\}\,\Box\, a<b} \qquad \dfrac{\dfrac{\mathfrak{F}[a]\Rightarrow\mathfrak{F}[a]}{\mathfrak{F}[a],\neg\mathfrak{F}[a]\Rightarrow}}{\mathfrak{F}[a],\neg\mathfrak{F}[a]\Rightarrow a=b}}{a\equiv 0, b\equiv 1, \mathfrak{F}[a], a\in\{0,1\}\,\Box\, a<b\rightarrow\neg\mathfrak{F}[a]\Rightarrow a=b}}{a\equiv 0, b\equiv 1, \mathfrak{F}[a], \bigwedge y\,(y\in\{0,1\}\,\Box\, y<b\rightarrow\neg\mathfrak{F}[y])\Rightarrow a=b}$$

Re 3.6iv. As for 3.6iii; left to the reader.
Re 3.6v. This is straightforward consequence of 3.6i–3.6iv.
Re 3.6vi. Employ 3.6v:

$$\dfrac{\dfrac{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[a] \qquad \dfrac{\dfrac{\mathfrak{C}[a],\mathfrak{C}[b]\Rightarrow a=b}{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[b]\rightarrow a=b}}{\mathfrak{C}[a]\Rightarrow\bigwedge y\,(\mathfrak{C}[y]\rightarrow a=y)}}{\dfrac{\mathfrak{C}[a]\Rightarrow\mathfrak{C}[a]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow a=y)}{\mathfrak{C}[a]\Rightarrow\bigvee x\,(\mathfrak{C}[x]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow x=y))}}}{\bigvee x\,\mathfrak{C}[x]\Rightarrow\bigvee x\,(\mathfrak{C}[x]\wedge\bigwedge y\,(\mathfrak{C}[y]\rightarrow x=y))}$$

Re 3.6vii. Straightforward in view of 3.6v; left to the reader. QED

THEOREM 3.7.

(3.7i) $\mathbf{L^iD}_\lambda\cup\{\bigvee x\,(\mathfrak{F}[x]\wedge\bigwedge y\,(\mathfrak{F}[y]\rightarrow x=y))\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\}\vdash\bot$

(3.7ii) $\mathbf{L^iD}_\lambda\cup\{\bigvee x\,\mathfrak{F}[x],\bigwedge z_1\bigwedge z_2\,(\mathfrak{F}[z_1]\,\Box\,\mathfrak{F}[z_2]\rightarrow z_1=z_2)$
$\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\}\vdash\bot$

Proof. The point is, of course, to find an appropriate $\mathfrak{F}$. That's what convention 3.5 has been designed for. In view of 3.6vi and 3.6vii, both, 3.7i and 3.7ii, essentially reduce to a form of 3.1, only with ι instead of ε:

$$\bigvee x\,\mathfrak{F}[x]\Rightarrow\mathfrak{F}[\iota x\,\mathfrak{F}[x]]\,,$$

where $\mathfrak{F}:\equiv *_1\in\{0,1\}\,\Box\,\phi\neq *_1\,\Box\,\bigwedge y\,(y\in\{0,1\}\,\Box\, y<*_1\rightarrow\neg(\phi\neq *_1))$ with ϕ being the fixed point satisfying $\phi=\iota x\mathfrak{F}[x]$. Since $\mathfrak{F}[\iota x\mathfrak{F}[x]]\Rightarrow\phi\neq\iota x\mathfrak{F}[x]$ is straightforward, one obtains

$$\bigvee x\,(x\in\{0,1\}\,\Box\,\phi\neq x\,\Box\,\bigwedge y\,(y\in\{0,1\}\,\Box\, y<x\rightarrow\neg(\phi\neq x))\Rightarrow\phi\neq\iota x\,\mathfrak{F}[x]\,.$$

The further procedure is essentially as for 3.4; the fixed point property provides $\phi \neq \iota x\, \mathfrak{F}[x] \Rightarrow$ and with some inversions of $\bigvee$ and $\square$ in the antecedent this gives:

$$0 \in \{0,1\}, \phi \neq 0, \bigwedge y\,(y \in \{0,1\} \square\, y < 0 \rightarrow \neg(\phi \neq 0)) \Rightarrow \ .$$

As before, one gets $\phi \neq 0 \Rightarrow$ and thereby $\phi \neq 1 \Rightarrow$ which, in turn, yields

$$\Rightarrow \bigwedge y\,(y \in \{0,1\} \square\, y < 1 \rightarrow \neg(\phi \neq 1))$$

as in the proof of 3.4; together with $\Rightarrow 1 \in \{0,1\}$ and

$$1 \in \{0,1\}, \phi \neq 1, \bigwedge y\,(y \in \{0,1\} \square\, y < 1 \rightarrow \neg(\phi \neq 1)) \Rightarrow$$

one obtains $\phi \neq 1 \Rightarrow$ by cut, hence a contradiction. QED

4. Addition 135g: An interpretation of $\lambda\beta$ in $\mathbf{L^iD^Z_\lambda}$

The availability of a notion of weak implication accounts for the possibility of expressing an arbitrary number of simple substitutions, *i.e.*, of substitutions achieved on the basis of $s = t$. This, in turn, makes it possibile to interpret $\lambda\beta$ in $\mathbf{L^iD^Z_\lambda}$.

Definitions 4.1. (1) $s \,\tilde{\in}\, t :\equiv \bigwedge y\,(s = y \supset y \in t)$.
(2) $\lambda^\dagger xy\, \mathfrak{F}[x,y] :\equiv \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda x\, \mathfrak{F}[x,y])$.
(3) LD-translation of λ-terms and wffs.

$$\text{(3.1)}\quad \mathfrak{A}[\|x\|^{\mathrm{LD}}] :\equiv \begin{cases} \mathfrak{A}[a] & \text{iff } x \text{ is not bound in } \mathfrak{A}\text{, and } a \text{ is the first in the list of free variables that does not occur in } \mathfrak{A} \\ \mathfrak{A}[x] & \text{otherwise} \end{cases}$$

(3.2) $\|\lambda x \boldsymbol{.} A\|^{\mathrm{LD}} :\equiv \lambda^\dagger xy\,(y = \|A\|^{\mathrm{LD}})$
(3.3) $\|AB\|^{\mathrm{LD}} :\equiv \lambda x \bigwedge y\,(\langle \|B\|^{\mathrm{LD}}, y\rangle \,\tilde{\in}\, \|A\|^{\mathrm{LD}} \rightarrow x \in y)$
(3.4) $\|A = B\|^{\mathrm{LD}} :\equiv \|A\|^{\mathrm{LD}} = \|B\|^{\mathrm{LD}}$

where y does not occur in A in clause (3.2), and neither x nor y occurs in AB in clause (3.3).

Convention 4.2. For the sake of simplicity, I shall write $\|A\|$ instead of $\|A\|^{\mathrm{LD}}$ for the remainder of this section.

EXAMPLES 4.3. The following examples are meant to give an idea of how λ-terms look under the LD-translation.

(1) $\|\lambda x\,\boldsymbol{.}\,x\| \equiv \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda(y = x))\,.$

(2) $\|\lambda xy\,\boldsymbol{.}\,x\| \equiv \lambda z_1 \bigwedge x_1 \bigwedge y_1\,(z_1 = \langle x_1, y_1\rangle \supset$
$x_1 \in \lambda(y_1 = \lambda z_2\,(\bigwedge x_2 \bigwedge y_2\,(z_2 = \langle x_2, y_2\rangle \supset x_2 \in \lambda(y_2 = x))))\,.$

(3) $\|\lambda xy\,\boldsymbol{.}\,xyy\| \equiv \lambda^{\dagger} x z_1\,(z_1 = \lambda^{\dagger} y z_2\,(z_2 =$
$\lambda x_1 \bigwedge y_1\,(\langle y, y_1\rangle \tilde{\in} \lambda x_2 \bigwedge y_2\,(\langle y, y_2\rangle \tilde{\in} x_2 \to x_2 \in y_2)) \to x_1 \in y_1))\,.$

PROPOSITION 4.4. *Inferences according to the following schemata are* $\mathbf{L^iD}_\lambda$*-derivable.*

(4.4i)
$$\frac{\Gamma \Rightarrow \mathfrak{F}[\|x\|]}{\Gamma \Rightarrow \bigwedge x\, \mathfrak{F}[x]}$$

if $\|x\|$ *does not occur in the lower sequent.*

(4.4ii)
$$\frac{\mathfrak{F}[\|A\|], \Gamma \Rightarrow C}{\bigwedge x\, \mathfrak{F}[x], \Gamma \Rightarrow C}$$

Proof. Re 4.4i. This is a straightforward consequence of clause (3.1) of definition 4.1.
Re 4.4ii. This is obvious in view of the fact that $\|A\|$ is a term in the language of $\mathbf{L^iD}_\lambda$. QED

PROPOSITION 4.5. *If the bound variable* x *does not occur in* $\mathfrak{F}$*, then*

$$\|B\| \in \lambda x\, \mathfrak{F}[x, \|A_1\|, \ldots, \|A_n\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|A_1[x/B]\|, \ldots, \|A_n[x/B]\|]\,.$$

is $\mathbf{L^iD}_\lambda$*-deducible.*

Proof. Employ an induction on the sum of the lengths of the A_i, where $i \in \{1, \ldots, n\}$. To save space, I confine myself to $n = 1$. Distinguish cases according to the clauses of definition 42.11 in [**15**], p. 502.
1. $A \equiv x$, *i.e.*, what has to be shown is

$$\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|x\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|x[x/B]\|]\,.$$

By definition 4.1 (3.1) and 42.11 (1) in [**15**], this amounts to showing

$$\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, x] \Leftrightarrow \mathfrak{F}[\|B\|, \|B\|]\,,$$

which is a straightforward application of λ-abstraction.

2. $A \equiv y$, where $x \not\equiv y$, *i.e.*, what has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|y\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|y[x/B]\|]\,.$$

By definition 4.1 (3.1) and 42.11 (2) in [**15**], this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|y\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|y\|]\,,$$

which is a straightforward application of λ-abstraction.
3. $A \equiv (C_1C_2)$, *i.e.*, what has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(C_1C_2)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(C_1C_2)[x/B]\|]\,.$$

By definition 42.11 (3) in [**15**], this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(C_1C_2)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(C_1[x/B]C_2[x/B])\|]\,,$$

which, by definition 4.1 (3.3), amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda x_1 \textstyle\bigwedge y_1 (\langle \|C_2\|, y_1\rangle \tilde{\in} \|C_1\| \to x_1 \in y_1)] \Leftrightarrow$$
$$\mathfrak{F}[\|B\|, \lambda x_1 \textstyle\bigwedge y_1 (\langle \|C_2[x/B]\|, y_1\rangle \tilde{\in} \|C_1[x/B]\|) \to x_1 \in y_1)]\,,$$

which, in turn, follows by the induction hypothesis.
4. $A \equiv (\lambda x . C)$. What has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda x . C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda x . C)[x/B]\|]\,.$$

By definition 42.11 (4), this amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda x . C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda x . C)\|]\,,$$

which is an immediate consequence of λ-abstraction.
5. $A \equiv (\lambda y . C)$ and $x \not\equiv y$. What has to be shown is

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y . C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|(\lambda y . C)[x/B]\|]\,.$$

Distinguish cases according to definition 42.11 in [**15**], clauses (5) and (6).
5.1. $y \notin FV(B)$ or $y \notin FV(C)$. By definition 42.11 (5) in [**15**], what has to be shown reduces to

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y . C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|\lambda y . C[x/B]\|]\,.$$

which, by definition 4.1 (3.2) amounts to showing

$$\mathbf{L^iD^Z_\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda^\dagger yz\,(z = \|C\|)] \Leftrightarrow \mathfrak{F}[\|B\|, \lambda^\dagger yz\,(z = \|C[x/B]\|)]\,,$$

which, in turn, follows by the inductions hypothesis.

5.2. $y \in FV(B)$ and $y \in FV(C)$. By definition 42.11 (6), what has to be shown reduces to

$$\mathbf{L^iD}^{\mathsf{Z}}_{\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \|(\lambda y \centerdot C)\|] \Leftrightarrow \mathfrak{F}[\|B\|, \|\lambda z \centerdot C[y/z][x/B]\|]$$

which, by definition 4.1 (3.4) can be reduced to

$$\mathbf{L^iD}^{\mathsf{Z}}_{\lambda} \vdash \|B\| \in \lambda x\, \mathfrak{F}[x, \lambda^{\dagger} yz\, (z = \|C\|)] \Leftrightarrow \mathfrak{F}[\|B\|, \lambda^{\dagger} zy_1\, (y_1 = \|C[y/z][x/B]\|)]$$

which, in turn, follows by the inductions hypothesis. QED

PROPOSITION 4.6. *Inferences according to the following schemata are* $\mathbf{L^iD}_{\lambda}$*-derivable.*

(4.6i)
$$\frac{\langle \|B_2\|, y\rangle \tilde{\in} \|A_2\|, \Gamma \Rightarrow \langle \|B_1\|, y\rangle \tilde{\in} \|A_1\|}{\Gamma \Rightarrow \|A_1 B_1\| = \|A_2 B_2\|}$$

where y is a free variable in the upper sequent, which does not occur in the lower sequent.

(4.6ii)
$$\frac{\Gamma \Rightarrow \|B\| = \|A\|}{\langle \|A\|, s\rangle \tilde{\in} t, \Gamma \Rightarrow \langle \|B\|, s\rangle \tilde{\in} t}$$

(4.6iii)
$$\frac{\Gamma \Rightarrow \|A[x/x_1]\| = \|B[x/x_1]\|}{\Gamma \Rightarrow \|\lambda x \centerdot A\| = \|\lambda x \centerdot B\|}$$

where $x_1 \notin FV(A)$ and $x_1 \notin FV(B)$.

Proof. Re 4.6i.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\langle \|B_2\|, b\rangle \tilde{\in} \|A_2\|, \Gamma \Rightarrow \langle \|B_1\|, b\rangle \tilde{\in} \|A_1\| \qquad a \in b \Rightarrow a \in b}{\langle \|B_1\|, b\rangle \tilde{\in} \|A_1\| \to a \in b, \langle \|B_2\|, b\rangle \tilde{\in} \|A_2\|, \Gamma \Rightarrow a \in b}}{\langle \|B_1\|, b\rangle \tilde{\in} \|A_1\| \to a \in b, \Gamma \Rightarrow \langle \|B_2\|, b\rangle \tilde{\in} \|A_2\| \to a \in b}}{\bigwedge y\, (\langle \|B_1\|, y\rangle \tilde{\in} \|A_1\| \to a \in y), \Gamma \Rightarrow \bigwedge y\, (\langle \|B_2\|, y\rangle \tilde{\in} \|A_2\| \to a \in y)}}{a \in \lambda x \bigwedge y\, (\langle \|B_1\|, y\rangle \tilde{\in} \|A_1\| \to x \in y), \Gamma \Rightarrow a \in \lambda x \bigwedge y\, (\langle \|B_2\|, y\rangle \tilde{\in} \|A_2\| \to x \in y)}}{\Gamma \Rightarrow a \in \|A_1 B_1\| \to a \in \|A_2 B_2\|}$$

Analogously for $\mathit{\Gamma} \Rightarrow a\in\|A_2B_2\| \to a\in\|A_1B_1\|$. Finish in a familiar way as follows:

$$\begin{array}{c}
\mathit{\Gamma} \Rightarrow a\in\|A_1B_1\| \to a\in\|A_2B_2\| \qquad\qquad \mathit{\Gamma} \Rightarrow a\in\|A_2B_2\| \to a\in\|A_1B_1\| \\ \hline
\mathit{\Gamma} \Rightarrow a\in\|A_1B_1\| \leftrightarrow a\in\|A_2B_2\| \\ \hline
\mathit{\Gamma} \Rightarrow \bigwedge x\,(x\in\|A_1B_1\| \leftrightarrow x\in\|A_2B_2\|)
\end{array}$$

Re 4.6ii. This is the point where the strength of Z-inferences is needed, and that in the inference marked by † which is according to 135.20vii in [**15**], p. 1847.

$$\begin{array}{c}
\mathit{\Gamma} \Rightarrow \|B\| = \|A\| \\ \hline
\mathit{\Gamma} \Rightarrow \langle\|B\|,s\rangle = \langle\|A\|,s\rangle \\ \hline
\langle\|B\|,s\rangle = b, \mathit{\Gamma} \Rightarrow \langle\|A\|,s\rangle = b \qquad\qquad b\in t \Rightarrow b\in t \\ \hline
\langle\|A\|,s\rangle = b \supset b\in t, \mathit{\Gamma} \Rightarrow \langle\|B\|,s\rangle = b \supset b\in t \quad \dagger \\ \hline
\bigwedge y\,(\langle\|A\|,s\rangle = y \supset y\in t), \mathit{\Gamma} \Rightarrow \langle\|B\|,s\rangle = b \supset b\in t \\ \hline
\bigwedge y\,(\langle\|A\|,s\rangle = y \supset y\in t), \mathit{\Gamma} \Rightarrow \bigwedge y\,(\langle\|B\|,s\rangle = y \supset y\in t) \\ \hline\hline
\langle\|A\|,s\rangle\,\tilde{\in}\,t, \mathit{\Gamma} \Rightarrow \langle\|B\|,s\rangle\,\tilde{\in}\,t
\end{array}$$

Re 4.6iii.

$$\begin{array}{c}
c = \langle\|x_1\|,b\rangle \Rightarrow c = \langle\|x_1\|,b\rangle \qquad
\begin{array}{c}
\mathit{\Gamma} \Rightarrow \|A[x/x_1]\| = \|B[x/x_1]\| \\ \hline
b = \|A[x/x_1]\|, \mathit{\Gamma} \Rightarrow b = \|B[x/x_1]\| \\ \hline\hline
\|x_1\|\in\lambda x\,(b = \|A\|), \mathit{\Gamma} \Rightarrow \|x_1\|\in\lambda x\,(b = \|B\|)
\end{array}\ 4.5 \\ \hline
c = \langle\|x_1\|,b\rangle \supset \|x_1\|\in\lambda x\,(b = \|A\|) \Rightarrow c = \langle\|x_1\|,b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|) \\ \hline\hline
\bigwedge x_1 \bigwedge y (c = \langle x_1,y\rangle \supset x_1\in\lambda x\,(y = \|A\|)) \Rightarrow c = \langle\|x_1\|,b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|) \\ \hline
c\in\|\lambda x\centerdot A\|, \mathit{\Gamma} \Rightarrow c = \langle\|x_1\|,b\rangle \supset \|x_1\|\in\lambda x\,(b = \|B\|) \\ \hline\hline
c\in\|\lambda x\centerdot A\|, \mathit{\Gamma} \Rightarrow \bigwedge x_1 \bigwedge y (c = \langle x_1,y\rangle \supset x_1\in\lambda x\,(y = \|B\|)) \\ \hline
c\in\|\lambda x\centerdot A\|, \mathit{\Gamma} \Rightarrow c\in\|\lambda x\centerdot B\| \\ \hline
\mathit{\Gamma} \Rightarrow c\in\|\lambda x\centerdot A\| \to c\in\|\lambda x\centerdot B\|
\end{array}$$

Continue as for 4.6i. QED

PROPOSITION 4.7. *If $y_1 \notin FV(B)$ and no variable bound in A is free in B, then there exists a natural number* n *such that*

$$\mathbf{L^iD}_\lambda \vdash \mathrm{n}[\|B\| = \|y_1\|] \Rightarrow \|A[x/B]\| = \|A[x/y_1]\|\,.$$

Proof by induction on the length of A. Distinguish cases according to the form of A. With the exception of the case that $A \equiv (C_1C_2)$, there is hardly any change to the proof of proposition 4.7 in the TOOLS, so I shall only treat that case.

$A \equiv (C_1C_2)$. As an immediate consequence of the induction hypothesis and proposition 4.6ii there is a natural number n_1 such that

$$\frac{\mathrm{n}_1[\|B\| = \|y_1\|] \Rightarrow \|C_2[x/B]\| = \|C_2[x/y_1]\|}{\langle \|C_2[x/y_1]\|, b\rangle \,\tilde{\in}\, \|C_1[x/y_1]\|, \mathrm{n}_1[\|B\| = \|y_1\|] \Rightarrow \langle \|C_2[x/B]\|, b\rangle \,\tilde{\in}\, \|C_1[x/y_1]\|}$$

is $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$-deducible. By the induction hypothesis there is also a natural number n_2 such that

$$\mathbf{L^iD}_{\lambda} \;\vdash\; \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|C_1[x/y_1]\| = \|C_1[x/B]\|\,.$$

This makes it possible to continue as follows, employing 4.6i:

$$\frac{\langle \|C_2[x/y_1]\|, b\rangle \,\tilde{\in}\, \|C_1[x/y_1]\|, \mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \langle \|C_2[x/B]\|, b\rangle \,\tilde{\in}\, \|C_1[x/B]\|}{\mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|C_1[x/B]C_2[x/B]\| = \|C_1[x/y_1]C_2[x/y_1]\|}$$

By definition 42.11 (3) in **[15]**, this is

$$\mathrm{n}_1 + \mathrm{n}_2[\|B\| = \|y_1\|] \Rightarrow \|(C_1C_2)[x/B]\| = \|(C_1C_2)[x/y_1]\|\,. \qquad \text{QED}$$

PROPOSITION 4.8. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(4.8i) $\quad \langle \|B\|, b\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\| \Rightarrow b = \|A[x/B]\|$

(4.8ii) $\quad s = \|A[x/B]\| \Rightarrow \langle \|B\|, s\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\|$

(4.8iii) $\quad \Rightarrow \langle \|B\|, \|A[x/B]\|\rangle \,\tilde{\in}\, \|\lambda x\,.\,A\|$

Proof. Re 4.8i.

$$\dfrac{\Rightarrow \langle \|B\|, b\rangle = \langle \|B\|, b\rangle \qquad \dfrac{b = \|A[x/B]\| \Rightarrow b = \|A[x/B]\|}{\|B\| \in \lambda x\,(b = \|A\|) \Rightarrow b = \|A[x/B]\|}}{\langle \|B\|, b\rangle = \langle \|B\|, b\rangle \supset \|B\| \in \lambda x\,(b = \|A\|) \Rightarrow b = \|A[x/B]\|}$$

$$\frac{\langle \|B\|, b\rangle = \langle \|B\|, b\rangle \supset \|B\| \in \lambda x\,(b = \|A\|) \Rightarrow b = \|A[x/B]\|}{\bigwedge x_1 \bigwedge y\,(\langle \|B\|, b\rangle = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}$$

$$\dfrac{\Rightarrow \langle \|B\|, b\rangle = \langle \|B\|, b\rangle \qquad \dfrac{\bigwedge x_1 \bigwedge y\,(\langle \|B\|, b\rangle = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}{\langle \|B\|, b\rangle \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}}{\dfrac{\langle \|B\|, b\rangle = \langle \|B\|, b\rangle \supset \langle \|B\|, b\rangle \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}{\dfrac{\bigwedge y_1\,(y_1 = \langle \|B\|, b\rangle \supset y_1 \in \lambda^{\dagger} xy\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}{\langle \|B\|, b\rangle \,\tilde{\in}\, \lambda z \bigwedge x_1 \bigwedge y\,(z = \langle x_1, y\rangle \supset x_1 \in \lambda x\,(y = \|A\|)) \Rightarrow b = \|A[x/B]\|}}}$$

Re 4.8ii. Employ 4.7, with z satisfying the necessary requirements.

$$
\begin{array}{c}
\mathrm{n}[\|B\| = \|z\|] \Rightarrow \|A[x/B]\| = \|A[x/z]\| \\
\hline
\mathrm{n}[\|B\| = \|z\|], s = \|A[x/B]\| \Rightarrow s = \|A[x/z]\| \\
\hline
\mathrm{n}[\|B\| = \|z\|], s = b, s = \|A[x/B]\| \Rightarrow b = \|A[x/z]\| \\
\hline
\mathrm{n}[\|B\| = \|z\|], s = b, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\, (b = \|A\|) \\
\hline
\mathrm{n}[\langle \|B\|, s\rangle = \langle \|z\|, b\rangle], s = b, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\, (b = \|A\|) \\
\hline
\mathrm{n}[\langle \|B\|, s\rangle = \langle \|z\|, b\rangle], \langle \|B\|, s\rangle = \langle \|z\|, b\rangle, s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\, (b = \|A\|) \\
\hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], \mathrm{n}+1[a = \langle \|z\|, b\rangle], s = \|A[x/B]\| \Rightarrow \|z\| \in \lambda x\, (b = \|A\|) \\
\hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow a = \langle \|z\|, b\rangle \supset \|z\| \in \lambda x\, (b = \|A\|) \\
\hline\hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow \bigwedge x_1 \bigwedge y\, (a = \langle x_1, y\rangle \supset x_1 \in \lambda x\, (y = \|A\|)) \\
\hline
\mathrm{n}+1[\langle \|B\|, s\rangle = a], s = \|A[x/B]\| \Rightarrow a \in \lambda^{\dagger} xy\, (y = \|A\|) \\
\hline
s = \|A[x/B]\| \Rightarrow a = \langle \|B\|, s\rangle \supset a \in \lambda^{\dagger} xy\, (y = \|A\|) \\
\hline
s = \|A[x/B]\| \Rightarrow \bigwedge y_1\, (y_1 = \langle \|B\|, s\rangle \supset y_1 \in \lambda^{\dagger} xy\, (y = \|A\|)) \\
\hline
s = \|A[x/B]\| \Rightarrow \langle \|B\|, s\rangle \,\tilde{\in}\, \lambda z \bigwedge x_1 \bigwedge y\, (z = \langle x_1, y\rangle \supset x_1 \in \lambda x\, (y = \|A\|))
\end{array}
$$

Re 4.8iii. This is an immediate consequence of 4.8ii. QED

REMARK 4.9. 4.6iii, 4.8i and 4.8ii above are the points where the notion of weak implication is really needed, more specifically, a notion of weak implication that satisfies the following schemata

$$
\frac{A, \ldots, A, \mathit{\Gamma} \Rightarrow B}{\mathit{\Gamma} \Rightarrow A \supset B}\,, \qquad \frac{A, \mathit{\Gamma} \Rightarrow B}{B \supset C, \mathit{\Gamma} \Rightarrow A \supset C}\,, \quad \text{and} \quad \frac{\Rightarrow A \qquad B, \mathit{\Gamma} \Rightarrow C}{A \supset B, \mathit{\Gamma} \Rightarrow C}\,.
$$

This concludes the listing of the relevant tools. I now begin with the translation of α-conversion.

PROPOSITION 4.10. *If* $y \notin FV(A)$, *then*

$$\mathbf{L^iD}_{\lambda} \vdash \Rightarrow \|\lambda x\,.\,A = \lambda y\,.\,A[x/y]\|\,.$$

Proof. What has to be shown is

$$\mathbf{L^iD}^{\mathrm{Z}}_{\lambda} \vdash \Rightarrow \|\lambda x\,.\,A\| = \|\lambda y\,.\,A[x/y]\|\,.$$

By proposition 4.6, it is sufficient to show

$$\mathbf{L^iD}^{\mathrm{Z}}_{\lambda} \vdash \Rightarrow \|A[x/x_1]\| = \|A[x/y][y/x_1]\|\,,$$

which is obvious in view of proposition 42.14i in [**15**], p. 502, establishing that $A[x/x_1]$ and $A[x/y][y/x_1]$ are actually identical in the sense of definition 42.11 in [**15**], if $y \notin FV(A)$. QED

I continue with the translation of β-conversion.

PROPOSITION 4.11. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(4.11i) $$s\in\|(\lambda x\,.\,A)B\| \Rightarrow s\in\|A[x/B]\|$$

(4.11ii) $$s\in\|A[x/B]\| \Rightarrow s\in\|(\lambda x\,.\,A)B\|$$

(4.11iii) $$\Rightarrow \|(\lambda x\,.\,A)B = A[x/B]\|$$

Proof. Re 4.11i. Employ 4.8iii.

$$\cfrac{\cfrac{\cfrac{\Rightarrow \langle\|B\|,\|A[x/B]\|\rangle\tilde{\in}\|\lambda x\,.\,A\| \qquad s\in\|A[x/B]\| \Rightarrow s\in\|A[x/B]\|}{\langle\|B\|,\|A[x/B]\|\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow s\in\|A[x/B]\| \Rightarrow s\in\|A[x/B]\|}}{\bigwedge y\,(\langle\|B\|,y\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow s\in y) \Rightarrow s\in\|A[x/B]\|}}{s\in\lambda x_1 \bigwedge y\,(\langle\|B\|,y\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow x_1\in y) \Rightarrow s\in\|A[x/B]\|}$$

Re 4.11ii.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{s\in\|A[x/B]\| \Rightarrow s\in\|A[x/B]\|}{s\in\|A[x/B]\|, b=\|A[x/B]\| \Rightarrow s\in b}}{s\in\|A[x/B]\|, \langle\|B\|,b\rangle\tilde{\in}\|\lambda x\,.\,A\|) \Rightarrow s\in b}\,{\scriptstyle 4.8\mathrm{i}}}{s\in\|A[x/B]\| \Rightarrow \langle\|B\|,b\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow s\in b}}{s\in\|A[x/B]\| \Rightarrow \bigwedge y\,(\langle\|B\|,y\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow s\in y)}}{s\in\|A[x/B]\| \Rightarrow s\in\lambda x_1 \bigwedge y\,(\langle\|B\|,y\rangle\tilde{\in}\|\lambda x\,.\,A\| \rightarrow x_1\in y)}$$

Re 4.11iii. This is a straightforward consequence of 4.11i and ii.
Re 4.11iv. This is a straightforward consequence of 4.11iii in view of the definition 4.1 (3.4). QED

PROPOSITION 4.12. *Inferences according to the following schemata are* $\mathbf{L^iD}_{\lambda}$*-derivable.*

(4.12i) $$\frac{\Rightarrow \|A=B\|}{\Rightarrow \|B=A\|}$$

(4.12ii)
$$\frac{\Rightarrow \|A = B\| \qquad \Rightarrow \|B = C\|}{\Rightarrow \|B = C\|}$$

(4.12iii)
$$\frac{\Rightarrow \|A = B\|}{\Rightarrow \|CA = CB\|}$$

Proof. *Re* 4.12i and 4.12ii. These are immediate consequences of the way = is defined in $\mathbf{L^iD}_\lambda$.
Re 4.12iii. This is a consequence of the inclusive character built into the definition of $\tilde{\in}$.

$$\cfrac{\cfrac{\Rightarrow \|A\| = \|B\|}{\langle \|B\|, y\rangle \,\tilde{\in}\, \|C\| \Rightarrow \langle \|A\|, y\rangle \,\tilde{\in}\, \|C\|}\;4.6\text{ii}}{\Rightarrow \|CA\| = \|CB\|}\;4.6\text{i}$$

QED

THEOREM 4.13. *If* $\lambda\beta \vdash A$, *then* $\mathbf{L^iD}^{\mathrm{Z}}_\lambda \vdash \Rightarrow \|A\|$.

Proof. 4.12i-iii are the translations of (σ), (τ) and (μ), respectively. QED

THEOREM 4.14. *Not every equation is* $\lambda\beta$*-deducible.*

Proof. This is an immediate consequence of the foregoing theorem 4.13 and the fact that $\|x = y\|$, *i.e.*, $a = b$, ist not an $\mathbf{L^iD}^{\mathrm{Z}}_\lambda$-deducible wff. QED

Discussion. In view of the smooth interpretation of logic in illative combinatory logic provided in [**11**],[13] the question will arise why is the translation provided here rather awkward in comparison? My answer is to draw attention to the notion of equality. The notion of equality in $\mathbf{L^iD}_\lambda$ (and, of course, $\mathbf{L^iD}^{\mathrm{Z}}_\lambda$) is provided by implication, conjunction, generalization (*i.e.*, illative notions) and elementhood in the usual way:

$$s = t :\equiv \bigwedge x\,((x \in s \to x \in t) \wedge (x \in t \to x \in s))\,.$$

This, however, does not agree too well with the introduction of = as a primitive relation in λ-calculus and combinatory logic. Consider the

[13] See, in particular, p. 587.

following situation:

$$\cfrac{CA \qquad \cfrac{\cfrac{A \to B \qquad B \to A}{A = B}}{CA = CB}\,(\mu)}{CB}\,Eq$$

which has already been shown to be incompatible in remark 42.64 (2) in [**15**], p. 517.

This doesn't seem to be too surprising if one considers the definition of $B \in A$ in [**11**], p. 587, as AB. The μ-inference together with the Eq-inference then reads:

$$\frac{A \in C \qquad A = B}{B \in C}$$

which just displays the characteristic feature of extensionality from a set theoretical perspective.

In other words, given the reading of equality in $\mathbf{L^iD^Z_\lambda}$, (μ) does actually provide a form of weak extensionality as considered in [**11**], p. 594, which has been shown to be incompatible with a formalized theory equivalent to $BCK\lambda\beta$ in U. [**15**], p. 517.

This, however, is compensated in the awkward definition of AB in the LD-translation as $\lambda x \bigwedge y\,(\langle B, y\rangle \mathrel{\tilde{\in}} A \to x \in y)$ by the somewhat "inclusive" notion $\tilde{\in}$.

Differently put: the system $BCK\lambda\beta$ from [**11**] becomes trivial, if something like

$$\frac{A \to B \qquad B \to A}{A = B}$$

is added as a basic rule of deduction (i.e., the premisses not depending on open assumptions).

5. Addition 137f. An approach to extending $\mathbf{L^iD^Z_\lambda}$ to accomodate nested double induction and recursion

This addition is still more of a suggestion than a fully worked out approach. The reason that it is included here is that it gives the idea of how I want to extend the approach begun with my **Z**-inferences to gain more deductive strength in systems of higher order logic without contraction.

5a. Introduction. Primitive recursion (or 1-recursion) is available in $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$ (in the sense that functions defined by primitive recursion from total functions can be explicitly defined and proven to be total), as shown in [**16**], but not so k-recursion for $k > 1$. The latter is readily concluded from a simple ordinal observation: a consistency proof for 2-recursion requires an induction up to $\omega^{\omega^{\omega}}$, while that of $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$ can be shown by an induction up to ω^{ω}. On the other hand, as I suggested at the end of [**16**], given a certain reinforced necessity operator obeying the rules

$$\frac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\boxed{\Box} A, \Gamma \Rightarrow C} \qquad \text{and} \qquad \frac{\Box^{\mathrm{n}} A \Rightarrow C}{\boxed{\Box} A \Rightarrow \boxed{\Box} C}\ ,$$

there is an easy way to overcome the difficulties. The present paper is dedicated to a way of introducing such a reinforced necessity operator $\boxed{\Box}$ *without* adding any new primitive symbols.[14]

Remark 5.1. Regarding the reduction step for the above rules: if the last part of a deduction has the form

$$\dfrac{\dfrac{\Box^{\mathrm{m}} B \Rightarrow A}{\boxed{\Box} B \Rightarrow \boxed{\Box} A} \qquad \dfrac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\boxed{\Box} A, \Gamma \Rightarrow C}}{\boxed{\Box} B, \Gamma \Rightarrow C}\ \clubsuit\ ,$$

then a deduction can be constructed as follow:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Box^{\mathrm{m}} B \Rightarrow A}}{\Box^{\mathrm{m+n}} B \Rightarrow \Box^{\mathrm{n}} A} \qquad \dfrac{\vdots}{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}}{\Box^{\mathrm{m+n}} B, \Gamma \Rightarrow C}\ \clubsuit}{\boxed{\Box} B, \Gamma \Rightarrow C}\ .$$

This may look pretty innocent. But since it is sufficient to provide 2-recursion, it will come no cheaper than by an induction up to $\omega^{\omega^{\omega}}$.

[14] No relief is to be expected from the introduction of function variables as promoted in, *e.g.*, [**9**] and [**8**] for the formulation of k-recursion for $k > 1$, simply because the problem in the present approach is not the *formulation* of an appropriate term, but the nested double induction required in the proof that it satisfies the criterion of a function: uniqueness of the value; and that problem prevails.

What I need is a way of quantifying, as it were, over necessity operators, and that in a way that allows a form of induction similar to that provided by $\breve{\mathbf{\Pi}}^{\circ}$ in [**14**]. This is what I am going to provide now.

My approach to providing sufficient deductive strength for proving 2-recursion is based heavily on [**14**] and [**16**] and is a further extension of the system $\mathbf{L^{i}D^{Z}_{\lambda}}$ presented in [**14**].

5b. $\mathbf{\Psi}_2$ and $\mathbf{Z}_2$. I begin by introducing a new kind of successor notion.

DEFINITIONS 5.2. (1) $s^{2\!I} :\equiv \lambda x \,\Box(x \in s)$ ("necessor", a kind of successor with regard to the necessity operator, a nec[essity-succ]essor).
(2) The set $\mathbf{\Psi}_2$ is defined inductively as follows:
(2.1) I is an element of $\mathbf{\Psi}_2$;
(2.1) If t is an element $\mathbf{\Psi}_2$, then so is $t^{2\!I}$.
(3) If n is a natural number, then the *corresponding* $\mathbf{\Psi}_2$*-element* is defined inductively as follows:
(3.1) I is the *corresponding* $\mathbf{\Psi}_2$*-element* to 0;
(3.2) If $\bar{\bar{n}}$ is the corresponding $\mathbf{\Psi}_2$-element to n, then $\bar{\bar{n}}^{2\!I}$ is the *corresponding* $\mathbf{\Psi}_2$-element to n$'$.

EXAMPLE 5.3. $I^{2\!I2\!I} \equiv \lambda x_1 \,\Box(x_1 \in \lambda x_2 \,\Box(x_2 \in I)) = \lambda x \,\Box\Box(x \in I)$.

CONVENTION 5.4. I shall use $\bar{\bar{m}}$ and $\bar{\bar{n}}$ as syntactic symbols for elements of $\mathbf{\Psi}_2$, possibly with index numbers.

REMARK 5.5. $\mathbf{\Psi}_2$ is the set $\{I, I^{2\!I}, I^{2\!I2\!I}, \ldots\}$, where

$$I^{2\!I} = \lambda x \,\Box(x = \mathcal{V}),$$
$$I^{2\!I2\!I} = \lambda x \,\Box\Box(x = \mathcal{V}),$$
$$\ldots .$$

Note, however, that this is equality, not identity! What is aimed at is, of course, this: $[A/\bar{\bar{n}}] \leftrightarrow \Box^{\mathrm{n}} A$.

PROPOSITION 5.6. *If $s \in \mathbf{\Psi}_2$, then there exists a natural number* n *such that s is the corresponding element of* n *in* $\mathbf{\Psi}_2$.

Proof. As for the case of $\mathbf{\Psi}$ in proposition 131.9 in [**15**], p. 1789, this is an immediate consequence of the definition of corresponding element. QED

PROPOSITION 5.7. *Sequents according to the following schemata are* $\mathbf{L^{i}D^{Z}_{\lambda}}$-*deducible.*

(5.7i) $[A/s^{2I}] \Leftrightarrow \Box[A/s]$

(5.7ii) $\Box([A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s]) \Rightarrow [A/s^{2I}] \mathbin{\scriptstyle\Box} [B/s^{2I}] \to [A \mathbin{\scriptstyle\Box} B/s^{2I}]$

(5.7iii) $[A/s^{2I}] \Rightarrow [A/s]$

(5.7iv) $[A/I^{2I}] \Leftrightarrow \Box A$

Proof. *Re* 5.7i. Immediate consequence of the abstraction rules.
Re 5.7ii.

$$\begin{array}{c}
[A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s], [A/s], [B/s] \Rightarrow [A \mathbin{\scriptstyle\Box} B/s] \\ \hline
\Box([A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s]), \Box[A/s], \Box[B/s] \Rightarrow \Box[A \mathbin{\scriptstyle\Box} B/s] \\ \hline\hline
\Box([A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s]), [A/s^{2I}], [B/s^{2I}] \Rightarrow [A \mathbin{\scriptstyle\Box} B/s^{2I}] \\ \hline
\Box([A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s]), [A/s^{2I}] \mathbin{\scriptstyle\Box} [B/s^{2I}] \Rightarrow [A \mathbin{\scriptstyle\Box} B/s^{2I}] \\ \hline
\Box([A/s] \mathbin{\scriptstyle\Box} [B/s] \to [A \mathbin{\scriptstyle\Box} B/s]) \Rightarrow [A/s^{2I}] \mathbin{\scriptstyle\Box} [B/s^{2I}] \to [A \mathbin{\scriptstyle\Box} B/s^{2I}]
\end{array}$$

Re 5.7iii.

$$\begin{array}{c}
\lambda A \in s \Rightarrow [A/s] \\ \hline
\Box(\lambda A \in s) \Rightarrow [A/s] \\ \hline
[A/s^{2I}] \Rightarrow [A/s]
\end{array}$$

Re 5.7iv. This is a straightforward consequence of 5.7i and 131.15i and ii in [**15**], p. 1792. QED

As for the case of $\mathbf{\Psi}$, the problem consists in capturing the informal notion $\mathbf{\Psi}_2$ on the formal level, and that in a way which provides for a form of induction. As in the case of $\mathbf{Z}$, the point is to find an application of self-reference (fixed-point construction) which creates, as it were, its own "successor", this time with regard to the necessity operator, *i.e.*, its own *necessor.* This is what the following definition aims at.

DEFINITION 5.8. $\breve{\gamma}_2[A] :\equiv \lambda x \Box(x \in x \mathbin{\scriptstyle\Box} A) \in \lambda x \Box(x \in x \mathbin{\scriptstyle\Box} A)$.

The next proposition lists a number of properties of $\breve{\gamma}_2$, somewhat paralleling proposition 132.5 in [**15**], p. 1804, concerning the case of $\breve{\gamma}$.

PROPOSITION 5.9. *Sequents according to the following schemata are* $\mathbf{L^{i}D^{Z}_{\lambda}}$-*deducible.*

(5.9i)	$\breve{\gamma}_2[A] \Leftrightarrow \Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} A)$
(5.9ii)	$\breve{\gamma}_2[A] \Rightarrow \Box A$
(5.9iii)	$\breve{\gamma}_2[A] \Rightarrow A$
(5.9iv)	$\breve{\gamma}_2[A] \Rightarrow \Box^{\mathrm{n}} A$
(5.9v)	$\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A]$
(5.9vi)	$\breve{\gamma}_2[A] \Rightarrow \Box^{\mathrm{n}}\breve{\gamma}_2[A]$
(5.9vii)	$\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} \Box A$
(5.9viii)	$\breve{\gamma}_2[A \wedge B] \Rightarrow \breve{\gamma}_2[A \wedge B] \mathbin{\scriptstyle\Box} \Box^{\mathrm{n}} A$
(5.9ix)	$\breve{\gamma}_2[A \wedge B] \Rightarrow \breve{\gamma}_2[A \wedge B] \mathbin{\scriptstyle\Box} \Box^{\mathrm{n}} B$

Proof. Re 5.9i. Straightforward consequence of the abstraction rules.
Re 5.9ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\breve{\gamma}_2[A], A \Rightarrow A}}{\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} A \Rightarrow A}}{\Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} A) \Rightarrow \Box A}}{\breve{\gamma}_2[A] \Rightarrow \Box A}$$

Re 5.9iii. Immediate consequence of 5.9ii.
Re 5.9iv. Repeat 5.9i.
Re 5.9v.

$$\dfrac{\dfrac{\dfrac{\dfrac{\breve{\gamma}_2[A] \Rightarrow \breve{\gamma}_2[A]}{\breve{\gamma}_2[A], A \Rightarrow \breve{\gamma}_2[A]}}{\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} A \Rightarrow \breve{\gamma}_2[A]}}{\Box(\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} A) \Rightarrow \Box\breve{\gamma}_2[A]}}{\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A]}$$

Re 5.9vi. Employ an induction on n, approaching along the line of 5.9v.

Re 5.9vii. Employ 5.9vi and 5.9ii:

$$\dfrac{\breve{\gamma}_2[A] \Rightarrow \Box\Box\breve{\gamma}_2[A] \qquad \dfrac{\dfrac{\Box\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \qquad \dfrac{\breve{\gamma}_2[A] \Rightarrow \Box A}{\Box\breve{\gamma}_2[A] \Rightarrow \Box A}}{\Box\breve{\gamma}_2[A], \Box\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} \Box A}}{\Box\Box\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} \Box A}}{\breve{\gamma}_2[A] \Rightarrow \Box\breve{\gamma}_2[A] \mathbin{\scriptstyle\Box} \Box A} \clubsuit$$

Re 5.9viii and 5.9ix. These are straightforward consequences of the foregoing results. QED

COROLLARY 5.10. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-derivable.*

(5.10i) $$\dfrac{A, \Gamma \Rightarrow C}{\breve{\gamma}_2[A], \Gamma \Rightarrow C}$$

(5.10ii) $$\dfrac{\Box A, \Gamma \Rightarrow C}{\breve{\gamma}_2[A], \Gamma \Rightarrow C}$$

(5.10iii) $$\dfrac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\breve{\gamma}_2[A], \Gamma \Rightarrow C}$$

(5.10iv) $$\dfrac{\breve{\gamma}_2[A \wedge B], \Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{\breve{\gamma}_2[A \wedge B], \Gamma \Rightarrow C}$$

(5.10v) $$\dfrac{\breve{\gamma}_2[A \wedge B], \Box^{\mathrm{n}} B, \Gamma \Rightarrow C}{\breve{\gamma}_2[A \wedge B], \Gamma \Rightarrow C}$$

(5.10vi) $$\dfrac{\Box^{\mathrm{n}}\breve{\gamma}_2[A], \Gamma \Rightarrow C}{\breve{\gamma}_2[A], \Gamma \Rightarrow C}$$

Next comes the definition of a term that is meant to do for the necessor ^{2I} what $\mathbf{Z}$ did for the verisection ^{I}.[15]

DEFINITION 5.11.
$\mathbf{Z}_2 :\equiv \lambda x \bigwedge y\, (\breve{\gamma}_2[I \in y \wedge \bigwedge z\, (\Box(z \in y) \rightarrow z^{2I} \in y)] \rightarrow x \in y)\,.$

15 As regards the term "verisection", cf. definition 131.5 on p. 1788 of [**15**].

Remark 5.12. This definition of $\mathbf{Z}_2$ is designed with an eye to a possible consistency proof somewhat along similar lines as that sketched in §133 of [**15**] for the case of $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$. At first sight, it may look as if the approach from [**14**] could be easily adapted from ^{I} to ^{2I}. This however, runs into trouble at the following point: while

$$\frac{\Rightarrow A}{\lambda A \in s \Rightarrow \lambda A \in s^{I}}$$

is perfectly $\mathbf{L^iD}_{\lambda}$-deducible, the following isn't $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$-deducible:

$$\frac{\Rightarrow A}{\lambda A \in s \Rightarrow \Box(\lambda A \in s)} \; .$$

A similar consideration applies to the employment of a fixed-point à la [**18**], *i.e.*, $\mathbf{Z}_2 = \lambda x \bigwedge y\, (I \in y \,\square\, \bigwedge z\, (z \in \mathbf{Z}_2 \rightarrow z^{2I} \in y) \rightarrow x \in y)$. It is with regard to this problem that the necessity operator is introduced in front of the "induction hypothesis", *i.e.*, the sub-formula $(z \in y)$ in $\mathbf{Z}_2$.

Proposition 5.13. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.13i) $\Rightarrow I \in \mathbf{Z}_2$

(5.13ii) $\Box(s \in \mathbf{Z}_2) \Rightarrow s^{2I} \in \mathbf{Z}_2$

Proof. Re 5.13i. Employ 5.9iii:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\breve{\gamma}_2[I \in b] \Rightarrow I \in b}{\breve{\gamma}_2[I \in b], \breve{\gamma}_2[\bigwedge z\, (\Box(z \in b) \rightarrow z^{2I} \in b)] \Rightarrow I \in b}}{\breve{\gamma}_2[I \in b] \,\square\, \breve{\gamma}_2[\bigwedge z\, (\Box(z \in b) \rightarrow z^{2I} \in b)] \Rightarrow I \in b}}{\Rightarrow \breve{\gamma}_2[I \in b] \,\square\, \breve{\gamma}_2[\bigwedge z\, (\Box(z \in b) \rightarrow z^{2I} \in b)] \rightarrow I \in b}}{\Rightarrow \bigwedge y\, (\breve{\gamma}_2[I \in y] \,\square\, \breve{\gamma}_2[\bigwedge z\, (\Box(z \in y) \rightarrow z^{2I} \in y)] \rightarrow I \in y)}}{\Rightarrow I \in \lambda x \bigwedge y\, (\breve{\gamma}_2[I \in y] \,\square\, \breve{\gamma}_2[\bigwedge z\, (\Box(z \in y) \rightarrow z^{2I} \in y)] \rightarrow x \in y)}$$

Re 5.13ii. Let $\mathfrak{Zet}_2 :\equiv \breve{\gamma}_2[I \in *_1 \wedge \bigwedge z\,(\Box(z \in *_1) \to z^{2I} \in *_1)]$:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\mathfrak{Zet}_2[b] \Rightarrow \mathfrak{Zet}_2[b] \qquad s \in b \Rightarrow s \in b \\ \hline
\mathfrak{Zet}_2[b] \to s \in b, \mathfrak{Zet}_2[b] \Rightarrow s \in b \\ \hline
\bigwedge y\,(\mathfrak{Zet}_2[y] \to s \in y), \mathfrak{Zet}_2[b] \Rightarrow s \in b \\ \hline
s \in \lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s \in y), \mathfrak{Zet}_2[b] \Rightarrow s \in b \\ \hline
\Box(s \in \lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s \in y)), \Box\mathfrak{Zet}_2[b] \Rightarrow \Box(s \in b) \\ \hline
\Box(s \in \lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to s \in y)), \mathfrak{Zet}_2[b] \Rightarrow \Box(s \in b)
\end{array} \; 5.10\text{vi} & s^{2I} \in b \Rightarrow s^{2I} \in b
\end{array} \\ \hline
\Box(s \in \mathbf{Z}_2), \mathfrak{Zet}_2[b], \Box(s \in b) \to s^{2I} \in b \Rightarrow s^{2I} \in b \\ \hline
\Box(s \in \mathbf{Z}_2), \mathfrak{Zet}_2[b], \bigwedge z\,(\Box(z \in b) \to z^{2I} \in b) \Rightarrow s^{2I} \in b \\ \hline
\Box(s \in \mathbf{Z}_2), \mathfrak{Zet}_2[b] \Rightarrow s^{2I} \in b \quad 5.10\text{v} \\ \hline
\Box(s \in \mathbf{Z}_2) \Rightarrow \mathfrak{Zet}_2[b] \to s^{2I} \in b \\ \hline
\Box(s \in \mathbf{Z}_2) \Rightarrow \bigwedge y\,(\mathfrak{Zet}_2[y] \to s^{2I} \in y) \\ \hline
\Box(s \in \mathbf{Z}_2) \Rightarrow s^{2I} \in \lambda x \bigwedge y\,(\mathfrak{Zet}_2[y] \to x \in y)
\end{array}
$$

QED

Remark 5.14. Notice that $\mathbf{L}^{\mathrm{i}}\mathbf{D}_{\lambda}^{\mathrm{Z}}$ is indeed required in the above deduction of 5.13ii.

5c. $\mathbf{Z}_2$-inferences and $\breve{\mathbf{\Pi}}^{\circ}_2$. As in the case of $\mathbf{Z}$, I shall next proceed to defining terms which provide for some form of proto-induction, in the present case a nested double one.

Definitions 5.15. (1) An inference according to the following schema is called a $\mathbf{Z}_2$-inference:

$$
\frac{\Gamma \Rightarrow s \in \mathbf{Z}_2 \qquad \Rightarrow A}{\Gamma \Rightarrow \lambda A \in s}\,.
$$

(2) The formalized theory $\mathbf{L}^{\mathrm{i}}\mathbf{D}_{\lambda}^{\mathrm{Z}_2}$ is defined as $\mathbf{L}^{\mathrm{i}}\mathbf{D}_{\lambda}^{\mathrm{Z}}$ plus all $\mathbf{Z}_2$-inferences.

Next comes the definition of $\breve{\mathbf{\Pi}}^{\circ}_2$. In what follows, I shall commonly write $[A/s]$ for $\lambda A \in s$, as I already did in [**14**].

DEFINITIONS 5.16. (1) $\mathfrak{P}\mathfrak{i}_2[s,t] :\equiv [I \in t/s] \wedge \bigwedge z\,[\Box(z \in t) \rightarrow z^{2I} \in t/s]\,.$
(2) $\breve{\mathbf{\Pi}}^{\circ}_2 :\equiv \lambda x\,(\Box(x \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\,(\mathfrak{P}\mathfrak{i}_2[x,y] \supset x \in y))\,.$

PROPOSITION 5.17. *Inferences according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.17i)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad\qquad \Box\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}$$

(5.17ii)
$$\frac{A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s] \Rightarrow [B/s]}$$

(5.17iii)
$$\frac{\Box A \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box[A/s] \Rightarrow [B/s]}$$

(5.17iv)
$$\frac{\Gamma \Rightarrow B}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [\Gamma/s] \Rightarrow [B/s]}$$

Proof. *Re* 5.17i. Let $\xi :\equiv \lambda x\,\mathfrak{F}[x]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mathfrak{F}[I]}{\Rightarrow I \in \lambda x\,\mathfrak{F}[x]}}{s \in \mathbf{Z}_2 \Rightarrow [I \in \lambda x\,\mathfrak{F}[x]/s]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{\Box(a \in \lambda x\,\mathfrak{F}[x]) \Rightarrow a^{2I} \in \lambda x\,\mathfrak{F}[x]}}{\Rightarrow \Box(a \in \lambda x\,\mathfrak{F}[x]) \rightarrow a^{2I} \in \lambda x\,\mathfrak{F}[x]}}{s \in \mathbf{Z}_2 \Rightarrow [\Box(a \in \xi) \rightarrow a^{2I} \in \xi/s]}}{s \in \mathbf{Z}_2 \Rightarrow \bigwedge z\,[\Box(z \in \xi) \rightarrow z^{2I} \in \xi/s]}}{s \in \mathbf{Z}_2 \Rightarrow [I \in \xi/s] \wedge \bigwedge z\,[\Box(z \in \xi) \rightarrow z^{2I} \in \xi/s]} \qquad \dfrac{\mathfrak{F}[s] \Rightarrow \mathfrak{F}[s]}{s \in \lambda x\,\mathfrak{F}[x] \Rightarrow \mathfrak{F}[s]}}{\Box(s \in \mathbf{Z}_2), [I \in \xi/s] \wedge \bigwedge z\,[\Box(z \in \xi) \rightarrow z^{2I} \in \xi/s] \supset s \in \xi \Rightarrow \mathfrak{F}[s]}}{\Box(s \in \mathbf{Z}_2), \bigwedge y\,([I \in y/s] \wedge \bigwedge z\,[\Box(z \in y) \rightarrow z^{2I} \in y/s] \supset s \in y) \Rightarrow \mathfrak{F}[s]}}{\Box(s \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\,([I \in y/s] \wedge \bigwedge z\,[\Box(z \in t) \rightarrow z^{2I} \in y/s] \supset s \in y) \Rightarrow \mathfrak{F}[s]}}{s \in \lambda x\,(\Box(s \in \mathbf{Z}_2) \mathbin{\scriptstyle\square} \bigwedge y\,([I \in y/s] \wedge \bigwedge z\,[\Box(z \in t) \rightarrow z^{2I} \in y/s] \supset s \in y)) \Rightarrow \mathfrak{F}[s]}$$

Re 5.17ii. Similar to 134.10ii in [**15**], p. 1830:

$$
\dfrac{
\dfrac{\dfrac{A \Rightarrow B}{[A/I] \Rightarrow [B/I]}}{\Rightarrow [A/I] \rightarrow [B/I]}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{[A/c] \Rightarrow [A/c] \qquad [B/c] \Rightarrow [B/c]}{[A/c] \rightarrow [B/c], [A/c] \Rightarrow [B/c]}}{\Box([A/c] \rightarrow [B/c]), \Box[A/c] \Rightarrow \Box([B/c])}}{\Box([A/c] \rightarrow [B/c]), [A/c^{2I}] \Rightarrow [B/c^{2I}]}}{\Box([A/c] \rightarrow [B/c]) \Rightarrow [A/c^{2I}] \rightarrow [B/c^{2I}]}
}{s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, [A/s] \Rightarrow [B/s]}\ 5.17\text{i}
$$

Re 5.17iii.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\Box[A/I] \Rightarrow \Box A \quad \Box A \Rightarrow B}{\Box[A/I \Rightarrow B}\clubsuit}{\Box[A/I] \Rightarrow [B/I]}}{\Rightarrow \Box[A/I \rightarrow [B/I]}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box[A/a] \Rightarrow \Box[A/a] \qquad [B/a] \Rightarrow [B/a]}{\Box[A/a] \rightarrow [B/a], \Box[A/a] \Rightarrow [B/a]}}{\Box[A/a] \rightarrow [B/a], [A/a^{2I}] \Rightarrow [B/a]}}{\Box(\Box[A/a] \rightarrow [B/a], \Box[A/a^{2I}] \Rightarrow \Box[B/a]}}{\Box(\Box[A/a] \rightarrow [B/a], \Box[A/a^{2I}] \Rightarrow [B/a^{2I}]}}{\Box(\Box[A/a] \rightarrow [B/a]) \Rightarrow \Box[A/a^{2I}] \rightarrow [B/a^{2I}]}
}{s \in \breve{\mathbf{\Pi}}^{\circ}_{2}, \Box[A/s] \Rightarrow [B/s]}\ 5.17\text{i}.
$$

Re 5.17iv. This is just a generalization of 5.17ii by taking the finite box-conjunction of the wffs of Γ. Left to the reader. QED

REMARK 5.18. There is something bordering on triviality in the "induction steps" of the proofs of 5.17ii–5.17iv, which is essentially due to the $\mathbf{L^{i}D}_{\lambda}$-deducibility of the sequent $\Box(\lambda A \in a) \Rightarrow \lambda A \in a^{2I}$:

$$
\dfrac{\Box(\lambda A \in a) \Rightarrow \Box(\lambda A \in a)}{\Box(\lambda A \in a) \Rightarrow \lambda A \in a^{2I}}\ .
$$

I suggest that this be seen in the context of the difference between a complete induction and a transinite induction. Every ordinal below ω can actually be reached by starting from 0 and adding 1, whereas with a transfinite ordinal one can only say that 0 can be reached by every descending chain.

PROPOSITION 5.19. *Sequents according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.19i)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \Box(s\in\mathbf{Z}_2)$
(5.19ii)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow s^{\mathfrak{I}}\in\mathbf{Z}_2$
(5.19iii)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [A/s] \Rightarrow A$
(5.19iv)	$\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_{2}$
(5.19v)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}}, t] \Rightarrow \mathfrak{P}\mathrm{i}_2[s, t^{\mathfrak{I}}]$
(5.19vi)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \Box(s\in t), \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}}, t] \Rightarrow s^{\mathfrak{I}}\in t$
(5.19vii)	$[\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_{2})]^{\cdot 2} \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_{2}$
(5.19viii)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_{2})$
(5.19ix)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \,\Box\, s\in\breve{\mathbf{\Pi}}^{\circ}_{2}$
(5.19x)	$s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_{2}$

Proof. Re 5.19i. Fairly immediate consequence of the definition; left to the reader.
Re 5.19ii. This is a straightforward consequence of 5.13ii and 5.19i.
Re 5.19iii.

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{[A/I] \Rightarrow A}}{\Rightarrow [A/I] \to A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(\lambda A\in c) \Rightarrow \Box[A/c]}{[A/c^{\mathfrak{I}}] \Rightarrow \Box[A/c]} \qquad \Box A \Rightarrow A}{\Box[A/c] \to \Box A, [A/c^{\mathfrak{I}}] \Rightarrow A}}{\Box([A/c] \to A), [A/c^{\mathfrak{I}}] \Rightarrow A}}{\Box([A/c] \to A) \Rightarrow [A/c^{\mathfrak{I}}] \to A}}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [A/s] \Rightarrow A}\,5.17\mathrm{i}\,.$$

Re 5.19iv. Obvious.

Re 5.19v. Inferences according to 5.17iii are marked by $*$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(I\in t)\Rightarrow\Box(I\in t)}{\Box(I\in t)\Rightarrow I\in t^{\mathfrak{A}}}\;*}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box[I\in t/s]\Rightarrow[I\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,[I\in t/s^{\mathfrak{A}}]\Rightarrow[I\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\mathfrak{Pi}_2[s^{\mathfrak{A}},t]\Rightarrow[I\in t^{\mathfrak{A}}/s]}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(c\in t)\Rightarrow\Box(c\in t)}{\Box(c\in t)\Rightarrow c\in t^{\mathfrak{A}}}\qquad c^{\mathfrak{A}}\in t\Rightarrow c^{\mathfrak{A}}\in t}{\Box(c\in t)\to c^{\mathfrak{A}}\in t,c\in t^{\mathfrak{A}}\Rightarrow c^{\mathfrak{A}}\in t}}{\Box(\Box(c\in t)\to c^{\mathfrak{A}}\in t),\Box(c\in t^{\mathfrak{A}})\Rightarrow\Box(c^{\mathfrak{A}}\in t)}}{\Box(\Box(c\in t)\to c^{\mathfrak{A}}\in t),\Box(c\in t^{\mathfrak{A}})\Rightarrow c^{\mathfrak{A}}\in t^{\mathfrak{A}}}}{\Box(\Box(c\in t)\to c^{\mathfrak{A}}\in t)\Rightarrow\Box(c\in t^{\mathfrak{A}})\to c^{\mathfrak{A}}\in t^{\mathfrak{A}}}\;*}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box[\Box(c\in t)\to c^{\mathfrak{A}}\in t/s]\Rightarrow[\Box(c\in t^{\mathfrak{A}})\to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,[\Box(c\in t)\to c^{\mathfrak{A}}\in t/s^{\mathfrak{A}}]\Rightarrow[\Box(c\in t^{\mathfrak{A}})\to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\bigwedge z[\Box(z\in t)\to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}]\Rightarrow[\Box(c\in t^{\mathfrak{A}})\to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\mathfrak{Pi}_2[s^{\mathfrak{A}},t]\Rightarrow[\Box(c\in t^{\mathfrak{A}})\to c^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\mathfrak{Pi}_2[s^{\mathfrak{A}},t]\Rightarrow\bigwedge z\,[\Box(z\in t^{\mathfrak{A}})\to z^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\mathfrak{Pi}_2[s^{\mathfrak{A}},t]\Rightarrow[I\in t^{\mathfrak{A}}/s]\wedge\bigwedge z\,[\Box(z\in t^{\mathfrak{A}})\to z^{\mathfrak{A}}\in t^{\mathfrak{A}}/s]}\;.$$

Re 5.19vi. Employ 5.19iii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(s\in t)\Rightarrow\Box(s\in t)\qquad s^{\mathfrak{A}}\in t\Rightarrow s^{\mathfrak{A}}\in t}{\Box(s\in t),\Box(s\in t)\to s^{\mathfrak{A}}\in t\Rightarrow s^{\mathfrak{A}}\in t}\;5.19\text{iii}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box(s\in t),[\Box(s\in t)\to s^{\mathfrak{A}}\in t/s]\Rightarrow s^{\mathfrak{A}}\in t}\;5.7\text{iii}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box(s\in t),[\Box(s\in t)\to s^{\mathfrak{A}}\in t/s^{\mathfrak{A}}]\Rightarrow s^{\mathfrak{A}}\in t}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box(s\in t),\bigwedge z\,[\Box(z\in t)\to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}]\Rightarrow s^{\mathfrak{A}}\in t}}{s\in\breve{\boldsymbol{\Pi}}^{\circ}_2,\Box(s\in t),[I\in t/s^{\mathfrak{A}}]\wedge\bigwedge z\,[\Box(z\in t)\to z^{\mathfrak{A}}\in t/s^{\mathfrak{A}}]\Rightarrow s^{\mathfrak{A}}\in t}\;.$$

Re 5.19vii. Employ 5.19ii, 5.19v, and 5.19vi. The inference marked by $*$ is according to 135.20vii in [**15**]:

$$
\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\mathbf{Z}_2}{\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \Box(s^{\mathfrak{I}}\in\mathbf{Z}_2)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \Rightarrow \mathfrak{P}\mathrm{i}_2[s,b^{\mathfrak{I}}] \quad \dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Box(s\in b), \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \Rightarrow s^{\mathfrak{I}}\in b}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, s\in b^{\mathfrak{I}}, \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \Rightarrow s^{\mathfrak{I}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2}, \mathfrak{P}\mathrm{i}_2[s,b^{\mathfrak{I}}] \supset s\in b^{\mathfrak{I}} \Rightarrow \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \supset s^{\mathfrak{I}}\in b}\,*}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2}, \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y] \supset s\in y) \Rightarrow \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \supset s^{\mathfrak{I}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2}, \Box(s^{\mathfrak{I}}\in\mathbf{Z}_2), \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y] \supset s\in y) \Rightarrow \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \supset s^{\mathfrak{I}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 2}, \Box(s^{\mathfrak{I}}\in\mathbf{Z}_2) \mathbin{\scriptstyle\Box} \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s,y] \supset s\in y) \Rightarrow \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \supset s^{\mathfrak{I}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 3} \Rightarrow \mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},b] \supset s^{\mathfrak{I}}\in b}}{[s\in\breve{\mathbf{\Pi}}^{\circ}_2]^{\cdot 3} \Rightarrow \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},y] \supset s^{\mathfrak{I}}\in y)}}{\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},y] \supset s^{\mathfrak{I}}\in y)}}{\dfrac{[\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2} \Rightarrow \Box(s^{\mathfrak{I}}\in\mathbf{Z}_2) \mathbin{\scriptstyle\Box} \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[s^{\mathfrak{I}},y] \supset s^{\mathfrak{I}}\in y)}{[\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2} \Rightarrow s^{\mathfrak{I}}\in\lambda x\,(\Box(x\in\mathbf{Z}_2) \mathbin{\scriptstyle\Box} \bigwedge y\,(\mathfrak{P}\mathrm{i}_2[x,y] \supset x\in y))}}\ .
$$

Re 5.19viii. Employ 5.19i and 5.19vii. Completely straightforward, but nevertheless, here is a deduction:

$$
\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2}{\Rightarrow \Box(I\in\breve{\mathbf{\Pi}}^{\circ}_2)} \qquad \dfrac{[\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2)]^{\cdot 2} \Rightarrow a^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2}{\Box\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \Box(a^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2)}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}\ 5.17\mathrm{i}\ .
$$

Re 5.19ix. This is an immediate consequence of 5.19viii.
Re 5.19x. This is a straightforward consequence of 5.19viii, ix, and vii:

$$
\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_2 \mathbin{\scriptstyle\Box} s\in\breve{\mathbf{\Pi}}^{\circ}_2 \qquad \dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \quad s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \mathbin{\scriptstyle\Box} s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \mathbin{\scriptstyle\Box} \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2)} \quad \dfrac{\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2), \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2}{\Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \mathbin{\scriptstyle\Box} \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \mathbin{\scriptstyle\Box} s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2}\,\clubsuit}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2}\,\clubsuit\ .
$$

QED

REMARK 5.20. Notice the strange detour in the deduction of $s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2$. I should very much like to call it a detour through infinity.

COROLLARY 5.21. *Inferences according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}}_{\lambda}$*-derivable.*

(5.21i)
$$
\dfrac{\mathit{\Gamma}, s^{\mathfrak{I}}\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathit{\Pi} \Rightarrow C}{\mathit{\Gamma}, s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathit{\Pi} \Rightarrow C}
$$

(5.21ii)
$$\frac{\Gamma, s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Pi, s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Theta \Rightarrow C}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \Gamma, \Pi, \Theta \Rightarrow C}$$

As in the case of $\breve{\mathbf{\Pi}}^{\circ}$, this provides for a form of “induction”.

THEOREM 5.22. *Inferences according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.22i)
$$\frac{\mathfrak{F}[I] \Rightarrow \qquad \mathfrak{F}[a^{2I}], \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \Box\mathfrak{F}[a]}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{F}[s] \Rightarrow C}$$

(5.22ii)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad \Box\mathfrak{F}[a], \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}$$

(5.22iii)
$$\frac{\Rightarrow \mathfrak{F}[I] \qquad [\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, \Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box\mathfrak{F}[s]}$$

Proof. Straightforward consequences of 5.17i and 5.21 in the usual way. *Re* 5.22ii. Employ 5.19x:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2 \quad \Rightarrow \mathfrak{F}[I]}{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[I]}}{\Rightarrow \Box(I\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[I])} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2}{a\in\breve{\mathbf{\Pi}}^{\circ}_2, \mathfrak{F}[a] \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a] \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2}}{\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a]) \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2} \quad \dfrac{\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2), \Box\,\mathfrak{F}[a] \Rightarrow \mathfrak{F}[a^{2I}]}{[\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a])]^{\cdot 2} \Rightarrow \mathfrak{F}[a^{2I}]}}{[\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a])]^{\cdot 3} \Rightarrow a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a^{2I}]}}{\Box\Box(a\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a]) \Rightarrow \Box(a^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[a^{2I}])}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box(s\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[s])}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_2 \,\square\, \mathfrak{F}[s]}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[s]}$$

Re 5.22iii.

$$
\dfrac{\dfrac{\Rightarrow \mathfrak{F}[I]}{\Rightarrow \Box\mathfrak{F}[I]} \qquad \dfrac{\dfrac{[\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, \Box(a \in \breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \mathfrak{F}[a^{2I}]}{[\Box\mathfrak{F}[a]]^{\cdot\mathrm{n}}, a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \mathfrak{F}[a^{2I}]}}{\Box\Box\mathfrak{F}[a], \Box(a \in \breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \Box\mathfrak{F}[a^{2I}]}}{s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \Box\mathfrak{F}[s]}
$$

QED

The next proposition somewhat corresponds to proposition 134.9 in [**15**], p. 1829.

PROPOSITION 5.23. *Sequents according to the following schemata are* $\mathbf{L^iD}^{\mathrm{Z}_2}_{\lambda}$*-deducible.*

(5.23i) $s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/s], [B/s] \Rightarrow [A \mathbin{\scriptstyle\Box} B/s]$

(5.23ii) $s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \to B/s], [A/s] \Rightarrow [B/s]$

(5.23iii) $s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \vee \neg A/s^{2I}], A \Rightarrow \Box A$

(5.23iv) $[A \vee \neg A/I], \Box A \Rightarrow [A/I]$

(5.23v) $s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \vee \neg A/s^{2I}], \Box A \Rightarrow [A/s^{2I}]$

(5.23vi) $s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \vee \neg A/s], \Box A \Rightarrow [A/s]$

Proof. Re 5.23i. In principle as for 5.23ii; left to the reader.
Re 5.23ii. I only show the "induction step":

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \to B/a] \Rightarrow [A \to B/a] \qquad [A/a] \Rightarrow [A/a]}{[A \to B/a], [A/a] \Rightarrow [A \to B/a] \mathbin{\scriptstyle\Box} [A/a]} \qquad [B/a] \Rightarrow [B/a]}{[A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a], [A \to B/a], [A/a] \Rightarrow [B/a]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), \Box[A \to B/a], \Box[A/a] \Rightarrow \Box[B/a]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), [A \to B/a^{2I}], [A/a^{2I}] \Rightarrow [B/a^{2I}]}\;5.7\mathrm{i}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]), [A \to B/a^{2I}] \mathbin{\scriptstyle\Box} [A/a^{2I}] \Rightarrow [B/a^{2I}]}}{\Box([A \to B/a] \mathbin{\scriptstyle\Box} [A/a] \to [B/a]) \Rightarrow [A \to B/a^{2I}] \mathbin{\scriptstyle\Box} [A/a^{2I}] \to [B/a^{2I}]}\ .
$$

Re 5.23iii. Employ 134.18i from [**15**], p. 1833:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box(A\vee\neg A), A \Rightarrow \Box A}{[A\vee\neg A/I^{2I}], A \Rightarrow \Box A}}{[A\vee\neg A/I^{2I}] \mathbin{\scriptstyle\square} A \Rightarrow \Box A}}{\Rightarrow [A\vee\neg A/I^{2I}] \mathbin{\scriptstyle\square} A \to \Box A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\vee\neg A/a^{2I}], A \Rightarrow [A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \qquad \Box A \Rightarrow \Box A}{[A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A, [A\vee\neg A/a^{2I}], A \Rightarrow \Box A}}{[A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A, \Box[A\vee\neg A/a^{2I}], A \Rightarrow \Box A}}{[A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A, [A\vee\neg A/a^{22I}], A \Rightarrow \Box A}}{[A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A, [A\vee\neg A/a^{22I}] \mathbin{\scriptstyle\square} A \Rightarrow \Box A}}{[A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A \Rightarrow [A\vee\neg A/a^{22I}] \mathbin{\scriptstyle\square} A \to \Box A}}{\Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} A \to \Box A) \Rightarrow [A\vee\neg A/a^{22I}] \mathbin{\scriptstyle\square} A \to \Box A}}{\dfrac{s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A\vee\neg A/s^{2I}] \mathbin{\scriptstyle\square} A \to \Box A}{s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/s^{2I}], A \Rightarrow \Box A}}\ .$$

Re 5.23iv. Employ 134.13i from [**15**], p. 1831:

$$\dfrac{\dfrac{\Box A \Rightarrow A}{\Box A \Rightarrow [A/I]}}{[A\vee\neg A/I], \Box A \Rightarrow [A/I]}\ .$$

Re 5.23v. I only show the "induction step"; employ 5.23iii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\vee\neg A/a^{2I}] \Rightarrow [A\vee\neg A/a^{2I}] \qquad a \in \breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{2I}], A \Rightarrow \Box A}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{2I}], [A\vee\neg A/a^{2I}], A \Rightarrow [A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A}}{\Box a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box[A\vee\neg A/a^{2I}], \Box A \Rightarrow \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A)}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box[A\vee\neg A/a^{2I}], \Box A \Rightarrow \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A)}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a^{22I}], \Box A \Rightarrow \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A) \qquad \Box[A/a^{2I}] \Rightarrow [A/a^{22I}]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A) \to \Box[A/a^{2I}], [A\vee\neg A/a^{22I}], \Box A \Rightarrow [A/a^{2I}]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A \to [A/a^{2I}]), [A\vee\neg A/a^{22I}] \mathbin{\scriptstyle\square} \Box A \Rightarrow [A/a^{2I}]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A \to [A/a^{2I}]), [A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A \Rightarrow [A/a^{2I}]}}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, \Box([A\vee\neg A/a^{2I}] \mathbin{\scriptstyle\square} \Box A \to [A/a]) \Rightarrow [A\vee\neg A/a^{22I}] \mathbin{\scriptstyle\square} \Box A \to [A/a^{2I}]}\ .$$

Re 5.23vi. Straightforward consequence of 5.23v and 5.23iv by 5.22ii.

QED

Remark 5.24. Apparently, 134.9iv in [**15**], p. 1829, doesn't survive in the form

$$s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \vee \neg A/s], A \Rightarrow [A/s]\,.$$

In order to see this, confront

$$A \vee \neg A, A \vee \neg A, A \Rightarrow A \mathbin{\scriptstyle\square} A \mathbin{\scriptstyle\square} A$$

which is obviously $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$-deducible, with

$$\square\square(A \vee \neg A), A \Rightarrow \square\square A$$

which, apparently, is not $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}}_{\lambda}$-deducible. But while 134.9iv survives in some form, at least, *viz.*, as 5.23vi, there doesn't seem to be anything corresponding to 134.9ii, *i.e.*, something like

$$s \in \breve{\mathbf{\Pi}}^{\circ}_2, [A \mathbin{\scriptstyle\square} B/s] \Rightarrow [A/s] \mathbin{\scriptstyle\square} [B/s]$$

perhaps. This can be seen from the following consideration: while

$$(A \mathbin{\scriptstyle\square} B) \mathbin{\scriptstyle\square} (A \mathbin{\scriptstyle\square} B) \Rightarrow (A \mathbin{\scriptstyle\square} A) \mathbin{\scriptstyle\square} (B \mathbin{\scriptstyle\square} B)$$

is $\mathbf{L}^{\mathbf{i}}\mathbf{D}_{\lambda}$-deducible,

$$\square(A \mathbin{\scriptstyle\square} B) \Rightarrow \square A \mathbin{\scriptstyle\square} \square B$$

is, apparently, not $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}}_{\lambda}$-deducible.[16] This may be taken to indicate that $\boxed{\scriptstyle 2}$ is not just a repetition of the same kind of necessity operator that is already available in $\square$.

5d. Applications. Just as $\breve{\mathbf{\Pi}}^{\circ}$ could be employed to define notions of necessity and weak implication, so can $\breve{\mathbf{\Pi}}^{\circ}_2$.

Definition 5.25. $\boxed{\scriptstyle 2}A :\equiv \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x])\,.$

The following proposition corresponds in an obvious way to proposition 134.13 in [**15**], p. 1831.

Proposition 5.26. *Sequents according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-deducible.*

(5.26i) $\boxed{\scriptstyle 2}A \Rightarrow A$

(5.26ii) $\boxed{\scriptstyle 2}A \Rightarrow \square A$

(5.26iii) $\boxed{\scriptstyle 2}A \Rightarrow \square^{\mathrm{n}} A$

(5.26iv) $\boxed{\scriptstyle 2}A \Rightarrow \boxed{\scriptstyle 2}\square A$

[16] Cf. remark 135.13 in [**15**], p. 1844.

(5.26v) $\boxed{2}\Box^{\mathrm{n}} A \Rightarrow \boxed{2}\Box^{\mathrm{n}'} A$

(5.26vi) $\boxed{2} A \Rightarrow \boxed{2}\Box^{\mathrm{n}} A$

(5.26vii) $\boxed{2} A, \boxed{2} B \Rightarrow \boxed{2}(A \mathbin{\scriptstyle\square} B)$

(5.26viii) $\boxed{2}(A \rightarrow B), \boxed{2} A \Rightarrow \boxed{2} B$

(5.26ix) $s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \boxed{2}(s \in \breve{\mathbf{\Pi}}^{\circ}_2)$

(5.26x) $s \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [s \in \breve{\mathbf{\Pi}}^{\circ}_2 / s]$

Proof. *Re* 5.26i. As for 134.13i in [**15**], p. 1831, only with $\breve{\mathbf{\Pi}}^{\circ}_2$ instead of $\breve{\mathbf{\Pi}}^{\circ}$.
Re 5.26ii and 5.26iii. In view of 5.26iv, these are left to the reader.
Re 5.26iv.

$$\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_2 \qquad \dfrac{\Box A \Rightarrow \Box A}{a \in \breve{\mathbf{\Pi}}^{\circ}_2, [A/a^{2I}] \Rightarrow [\Box A/a]}\ \text{5.17iii}}{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}$$

$$\dfrac{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}\ \text{5.21ii}$$

$$\dfrac{a^{2I} \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a^{2I}], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}$$

$$\dfrac{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [\Box A/a]}{\dfrac{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]) \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [\Box A/a]}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]) \Rightarrow \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [\Box A/x])}}$$

Re 5.26v. This is just 5.26iv, only with $\Box^{\mathrm{n}} A$ being substituted for A.
Re 5.26vi. Employ an induction on n, based on 5.26v and 5.26vi.
Re 5.26vii. Employ 5.23i:

$$\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_2 \qquad [A/a], [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}$$

$$\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2, a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}\ \text{5.21ii}$$

$$\dfrac{a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/a], a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}$$

$$\dfrac{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/x]), a \in \breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [A \mathbin{\scriptstyle\square} B/a]}{\dfrac{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/x]) \Rightarrow a \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A \mathbin{\scriptstyle\square} B/x]}{\bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x]), \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [B/x]) \Rightarrow \bigwedge x\,(x \in \breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A \mathbin{\scriptstyle\square} B/x])}}$$

Re 5.26viii. Essentially, what has to be shown is that

$$\Rightarrow [A \rightarrow B/I] \mathbin{\scriptstyle\square} [A/I] \rightarrow [B/I], \text{ and}$$

$$\Box([A \rightarrow B/a] \mathbin{\scriptstyle\square} [A/a] \rightarrow [B/a]) \Rightarrow [A \rightarrow B/a^{2I}] \mathbin{\scriptstyle\square} [A/a^{2I}] \rightarrow [B/a^{2I}]$$

are $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$-deducible. The first one is completely straightforward. As regards the second one, employ 5.23ii and proceed as for 5.26vii:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \qquad [A\to B/a],[A/a],a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [B/a]}{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A\to B/a], a\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A/a], a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [B/a]}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A\to B/a], a\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A/a], a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [B/a]}\,5.21\text{ii}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A/x]), a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [B/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A/x]) \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [B/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A\to B/x]), \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [A/x]) \Rightarrow \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [B/x])}$$

Re 5.26ix. Employ 5.19iv and 5.19x;

$$\dfrac{\dfrac{\Rightarrow I\in\breve{\mathbf{\Pi}}^{\circ}_2}{\Rightarrow \boxed{2}(I\in\breve{\mathbf{\Pi}}^{\circ}_2)} \qquad \dfrac{c\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow c^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2}{\boxed{2}(c\in\breve{\mathbf{\Pi}}^{\circ}_2) \Rightarrow \boxed{2}(c^{2I}\in\breve{\mathbf{\Pi}}^{\circ}_2)}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \boxed{2}(s\in\breve{\mathbf{\Pi}}^{\circ}_2)}$$

Re 5.26x. Employ 5.26ix:

$$\dfrac{\dfrac{\dfrac{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [s\in\breve{\mathbf{\Pi}}^{\circ}_2/x])}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow s\in\breve{\mathbf{\Pi}}^{\circ}_2 \to [s\in\breve{\mathbf{\Pi}}^{\circ}_2/s]}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [s\in\breve{\mathbf{\Pi}}^{\circ}_2/s]}}{s\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow [s\in\breve{\mathbf{\Pi}}^{\circ}_2/s]}$$

QED

PROPOSITION 5.27. *Inferences according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.27i) $$\dfrac{\Rightarrow A}{\Rightarrow \Box A}$$

(5.27ii) $$\dfrac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow B}{\boxed{2} A, \Gamma \Rightarrow B}$$

(5.27iii) $$\dfrac{\Box A \Rightarrow B}{\boxed{2} A \Rightarrow \boxed{2} B}$$

(5.27iv) $$\dfrac{\Box^{\mathrm{n}} A \Rightarrow B}{\boxed{2} A \Rightarrow \boxed{2} B}$$

$$\text{(5.27v)}\qquad \frac{\Box^{n}\Gamma \Rightarrow C}{\boxed{2}\Gamma \Rightarrow \boxed{2}C}$$

$$\text{(5.27vi)}\qquad \frac{\Gamma \Rightarrow A}{s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \boxed{2}\Gamma \Rightarrow [A/s]}$$

Proof. Re 5.27i.

$$\dfrac{\dfrac{\Rightarrow [A/I] \qquad \dfrac{\Box[A/c] \Rightarrow \Box[A/c]}{\Box[A/c] \Rightarrow [A/c^{2I}]}}{a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [A/a]}}{\dfrac{\Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/a]}{\Rightarrow \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [A/x])}}$$

Re 5.27ii–5.27vi. These are all fairly straightforward consequences of the results from 5.26 by means of 5.27i. I only show 5.27iv as an example. Employ 5.26vi and 5.26viii:

$$\dfrac{\boxed{2}A \Rightarrow \boxed{2}\Box^{n}A \qquad \dfrac{\dfrac{\dfrac{\Box^{n}A \Rightarrow B}{\Rightarrow \Box^{n}A \rightarrow B}}{\Rightarrow \boxed{2}(\Box^{n}A \rightarrow B)}\,5.27\text{i} \qquad \boxed{2}(\Box^{n}A \rightarrow B), \boxed{2}\Box^{n}A \Rightarrow \boxed{2}B}{\boxed{2}\Box^{n}A \Rightarrow \boxed{2}B}\,\clubsuit}{\boxed{2}A \Rightarrow \boxed{2}B}\,\clubsuit\,.$$

QED

Remark 5.28. In view of 5.27ii and 5.27iv above, we can now say $\boxed{2}$ realizes the intention of $\boxed{\Box}$. The new symbol is chosen to allow a further development of the hierarchy: $\boxed{3}$, $\boxed{4}$, *etc.*, with $\boxed{1}$, of course, being $\Box$.

Proposition 5.29. *Sequents according to the following schemata are* $\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda}$*-deducible.*

(5.29i) $\boxed{2}(A \vee \neg A), \Box A \Rightarrow \boxed{2}A$

(5.29ii) $\boxed{2}(A \vee \neg A), \boxed{2}A \rightarrow B \Rightarrow \Box A \rightarrow B$

Proof. *Re* 5.29i. Employ 5.23vi:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \qquad a\in\breve{\mathbf{\Pi}}^{\circ}_2, [A\vee\neg A/a], \Box A \Rightarrow [A/a]}{a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow [A/a]}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2, a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow [A/a]}\,5.21\text{ii}}{a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/a], \Box A \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/x]), \Box A \Rightarrow a\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/a]}}{\bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A\vee\neg A/x]), \Box A \Rightarrow \bigwedge x\,(x\in\breve{\mathbf{\Pi}}^{\circ}_2 \rightarrow [A/x])}\;.$$

Re 5.29ii. As for 134.18ii in [**15**], p. 1833, this is a straightforward consequence of the foregoing result, in this case 5.29i:

$$\dfrac{\dfrac{\boxed{2}(A\vee\neg A), A \Rightarrow \boxed{2}A \qquad B \Rightarrow B}{\boxed{2}(A\vee\neg A), \boxed{2}A \rightarrow B, \Box A \Rightarrow B}}{\boxed{2}(A\vee\neg A), \boxed{2}A \rightarrow B \Rightarrow \Box A \rightarrow B}\;.$$

QED

With the notion of $\boxed{2}$ available, a form of induction with side-wffs, *i.e.*, induction under assumptions, can be established for $\breve{\mathbf{\Pi}}^{\circ}_2$-induction, just as in the case of $\breve{\mathbf{\Pi}}^{\circ}$ and $\Box$.

Proposition 5.30. *Inferences according to the following schema are* $\mathbf{L^{i}D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

$$\dfrac{\Gamma \Rightarrow \mathfrak{F}[I] \qquad \Box\mathfrak{F}[a], a\in\breve{\mathbf{\Pi}}^{\circ}_2, \Gamma \Rightarrow \mathfrak{F}[a^{2I}]}{s\in\breve{\mathbf{\Pi}}^{\circ}_2, \boxed{2}\Gamma \Rightarrow \mathfrak{F}[s]}$$

Proof. Let $\xi :\equiv \lambda x\, \mathfrak{F}[x]$. Employ 5.19x:

$$
\begin{array}{c}
\square\,\mathfrak{F}[a], \mathit{\Gamma}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[a^{\mathit{2I}}] \\
\hline\hline
\square\,(a\in\xi), \mathit{\Gamma}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow a^{\mathit{2I}}\in\xi \\
\hline
\mathit{\Gamma}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \square\,(a\in\xi) \rightarrow (a^{\mathit{2I}}\in\xi) \\
\hline
\text{5.17iv} \\
\mathit{\Gamma} \Rightarrow \mathfrak{F}[I] \qquad s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s], [s\in\breve{\mathbf{\Pi}}^{\circ}_{2}/s] \Rightarrow [\square\,(a\in\xi) \rightarrow (a^{\mathit{2I}}\in\xi)/s] \\
\hline
\text{5.26x} \\
\mathit{\Gamma} \Rightarrow I\in\xi \qquad s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s], s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow [\square\,(a\in\xi) \rightarrow (a^{\mathit{2I}}\in\xi)/s] \\
\hline
\text{5.21ii} \\
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s] \Rightarrow [I\in\xi/s] \qquad s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s] \Rightarrow [\square\,(a\in\xi) \rightarrow (a^{\mathit{2I}}\in\xi)/s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s] \Rightarrow \bigwedge z\,[\square\,(z\in\xi) \rightarrow (z^{\mathit{2I}}\in\xi)/s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s] \Rightarrow \mathfrak{P}\mathfrak{i}_2[s,\xi] \qquad s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
\square(s\in\breve{\mathbf{\Pi}}^{\circ}_{2}), \square[\mathit{\Gamma}/s], \mathfrak{P}\mathfrak{i}_2[s,\xi] \supset s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
\text{5.7i} \\
\square(s\in\breve{\mathbf{\Pi}}^{\circ}_{2}), [\mathit{\Gamma}/s^{\mathit{2I}}], \mathfrak{P}\mathfrak{i}_2[s,\xi] \supset s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
\text{5.26ix} \\
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s^{\mathit{2I}}], \mathfrak{P}\mathfrak{i}_2[s,\xi] \supset s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s^{\mathit{2I}}], \square(s\in\mathbf{Z}_2), \mathfrak{P}\mathfrak{i}_2[s,\xi] \supset s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s^{\mathit{2I}}], \square(s\in\mathbf{Z}_2) \mathbin{\scriptstyle\square} \mathfrak{P}\mathfrak{i}_2[s,\xi] \supset s\in\xi \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow s^{\mathit{2I}}\in\breve{\mathbf{\Pi}}^{\circ}_{2} \qquad s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, [\mathit{\Gamma}/s^{\mathit{2I}}], s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, s^{\mathit{2I}}\in\breve{\mathbf{\Pi}}^{\circ}_{2} \rightarrow [\mathit{\Gamma}/s^{\mathit{2I}}], s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \boxed{2}\mathit{\Gamma}, s\in\breve{\mathbf{\Pi}}^{\circ}_{2} \Rightarrow \mathfrak{F}[s] \\
\hline
s\in\breve{\mathbf{\Pi}}^{\circ}_{2}, \boxed{2}\mathit{\Gamma} \Rightarrow \mathfrak{F}[s]
\end{array}
$$

QED

As in the case of $\breve{\mathbf{\Pi}}^{\circ}$, it is useful to introduce some form of an inclusive version of $\breve{\mathbf{\Pi}}^{\circ}_{2}$.

DEFINITION 5.31. $\breve{\mathbf{\Pi}}_2 :\equiv \lambda x \bigvee y\,(\boxed{2}(y = x) \mathbin{\scriptstyle\square} y\in\breve{\mathbf{\Pi}}^{\circ}_{2})$.

PROPOSITION 5.32. *Inferences according to the following schemata are* $\mathbf{L}^{\mathbf{i}}\mathbf{D}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

(5.32i)
$$\frac{A \Rightarrow B}{s\in\breve{\mathbf{\Pi}}_2, [A/s] \Rightarrow [B/s]}$$

(5.32ii)
$$\frac{\square A \Rightarrow B}{s\in\breve{\mathbf{\Pi}}_2, \square[A/s] \Rightarrow [B/s]}$$

(5.32iii)
$$\frac{\Gamma \Rightarrow B}{s \in \breve{\mathbf{\Pi}}_2, [\Gamma/s] \Rightarrow [B/s]}$$

Proof. Essentially consequences of the corresponding proposition 5.17 for the exclusive case. I shall only show 5.32i as an example.
Re 5.32i. Employ 5.17ii:

$$\begin{array}{c}
b \in \breve{\mathbf{\Pi}}^\circ_2, [A/b] \Rightarrow [B/b] \\
\hline\hline
b = s, b = s, b \in \breve{\mathbf{\Pi}}^\circ_2, [A/s] \Rightarrow [B/s] \\
\hline
\Box(b = s), b \in \breve{\mathbf{\Pi}}^\circ_2, [A/s] \Rightarrow [B/s] \\
\hline
\boxed{2}(b = s), b \in \breve{\mathbf{\Pi}}^\circ_2, [A/s] \Rightarrow [B/s] \\
\hline
\boxed{2}(b = s) \,\square\, b \in \breve{\mathbf{\Pi}}^\circ_2, [A/s] \Rightarrow [B/s] \\
\hline
\bigvee y \, (\boxed{2}(y = s) \,\square\, y \in \breve{\mathbf{\Pi}}^\circ_2), [A/s] \Rightarrow [B/s] \\
\hline
s \in \breve{\mathbf{\Pi}}_2, [A/s] \Rightarrow [B/s]
\end{array}$$

QED

As in the case of $\breve{\mathbf{\Pi}}$, this gives rise to a notion of "weak" implication.

DEFINITION 5.33. $A \circledcirc_2 B :\equiv \bigvee x \, (x \in \breve{\mathbf{\Pi}}_2 \,\square\, ([A/x] \rightarrow B))$.

PROPOSITION 5.34. *Inferences according to the following schemata are* $\mathbf{L^iD^Z_\lambda}$*-deducible.*

(5.34i)
$$\frac{[A]^{\cdot \mathrm{n}}, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \circledcirc_2 B}$$

(5.34ii)
$$\frac{\Box^{\mathrm{n}} A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \circledcirc_2 B}$$

(5.34iii)
$$\frac{\Gamma \Rightarrow A \qquad \Pi \Rightarrow A \circledcirc_2 B}{\boxed{2}\Gamma, \Pi \Rightarrow B}$$

(5.34iv)
$$\frac{\Gamma \Rightarrow A \qquad B, \Pi \Rightarrow C}{A \circledcirc_2 B, \boxed{2}\Gamma, \Pi \Rightarrow C}$$

(5.34v)
$$\frac{(A \rightarrow B) \Rightarrow (C_1 \rightarrow (\ldots \rightarrow (C_{\mathrm{n}} \rightarrow B)\ldots))}{(A \circledcirc_2 B) \Rightarrow (C_1 \circledcirc_2 (\ldots \circledcirc_2 (C_{\mathrm{n}} \circledcirc_2 B)\ldots))}$$

(5.34vi)
$$\frac{A_2, \Gamma \Rightarrow A_1 \qquad B_1, \Box^{\mathrm{n}} A_2, \Pi \Rightarrow B_2}{A_1 \circledcirc_2 B_1, \Gamma, \Pi \Rightarrow A_2 \circledcirc_2 B_2}$$

(5.34vii)
$$\frac{B, \Box^{\mathrm{n}} A, \Gamma \Rightarrow C}{A \Supset B, \Gamma \Rightarrow A \Supset C}$$

Proof. In view of the similarity to the case of $\supset$ in [**15**], propositions 135.17, 135.20 and 135.22, I leave the proof to the reader. QED

REMARK 5.35. In view of remark 5.24 above, inferences according to the following schema
$$\frac{\boxed{2} A, \Gamma \Rightarrow C}{\boxed{2}(A \vee \neg A), A, \Gamma \Rightarrow \boxed{2} C}$$
cannot be expected to be generally $\mathbf{L^iD}^{\mathrm{Z}_2}_{\lambda}$-derivable.

I now turn to the reason why I have gone to all the trouble with the notion of $\boxed{2}$: nested double induction.

PROPOSITION 5.36. *Inferences according to the following schema are* $\mathbf{L^iD}^{\mathrm{Z}_2}_{\lambda}$*-derivable.*

$$\frac{\begin{array}{l}\Gamma, b \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[0,b] \\ \qquad \bigwedge^\circ y\, \mathfrak{F}[a,y], \Pi, a \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[a',0] \\ \qquad\qquad \bigwedge^\circ y\, \mathfrak{F}[a,y], \mathfrak{F}[a',b], a \in \mathbf{N}^\circ, b \in \mathbf{N}^\circ, \Xi \Rightarrow \mathfrak{F}[a',b']\end{array}}{s \in \mathbf{N}^\circ, t \in \mathbf{N}^\circ, \boxed{2}\Gamma, \Box\boxed{2}\Pi, \Box\boxed{2}\Xi \Rightarrow \mathfrak{F}[s,t]}$$

Proof. The inference marked by $*_1$ is somewhat (give or take some weakenings) according to 136.11ii, and that marked by $*_2$ is according to 136.11iii in [**15**], p. 1863.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, b \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[0,b]}{\Gamma \Rightarrow \bigwedge^\circ y\, \mathfrak{F}[0,y]}}{\boxed{2}\Gamma \Rightarrow \boxed{2} \bigwedge^\circ y\, \mathfrak{F}[0,y]} \qquad \cfrac{\cfrac{\cfrac{\bigwedge^\circ y\, \mathfrak{F}[a,y], \Pi, a \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[a',0] \quad \bigwedge^\circ y\, \mathfrak{F}[a,y], \mathfrak{F}[a',b], a \in \mathbf{N}^\circ, b \in \mathbf{N}^\circ, \Xi \Rightarrow \mathfrak{F}[a',b']}{\Box \bigwedge^\circ y\, \mathfrak{F}[a,y], \Box\Pi, \Box\Xi, c \in \mathbf{N}^\circ \Rightarrow \mathfrak{F}[a',c]}\,*_1}{\Box \bigwedge^\circ y\, \mathfrak{F}[a,y], \Box\Pi, \Box\Xi \Rightarrow \bigwedge^\circ y\, \mathfrak{F}[a',y]}}{\boxed{2} \bigwedge^\circ y\, \mathfrak{F}[a,y], \boxed{2}\Pi, \boxed{2}\Xi \Rightarrow \boxed{2} \bigwedge^\circ y\, \mathfrak{F}[a',y]}}{s \in \mathbf{N}^\circ, \boxed{2}\Gamma, \Box\boxed{2}\Pi, \Box\boxed{2}\Xi \Rightarrow \boxed{2} \bigwedge^\circ y\, \mathfrak{F}[s,y]}\,*_2}{\cfrac{s \in \mathbf{N}^\circ, \boxed{2}\Gamma, \Box\boxed{2}\Pi, \Box\boxed{2}\Xi \Rightarrow \bigwedge^\circ y\, \mathfrak{F}[s,y]}{s \in \mathbf{N}^\circ, t \in \mathbf{N}^\circ, \boxed{2}\Gamma, \Box\boxed{2}\Pi, \Box\boxed{2}\Xi \Rightarrow \mathfrak{F}[s,t]}}$$

QED

6. General Corrections[17]

Note. This list does not necessarily cover obvious typos or silly little grammatical mistakes and it is far from being complete. I you find a mistake, please drop me a note at `uwe.petersen@asfpg.de`, and I promise that you will get a mention in the next list.

p. 30, line 4: replace "$\mathrm{f(z) = y}$" by "$\mathrm{g(z) = y}$".
p. 45, first line: replace "additive numbers" by "principal numbers".
— line 16 (DEFINITION 4.26), before "*multiplicative principal number*" insert "(2) An ordinal number α is called a".

p. 65, line 8 from the bottom (HISTORICAL NOTE 8.8), replace "Dedekind [1887]" by "Dedekind [1888]".

p. 75, line 11 from the bottom, replace "$\mathrm{f_k(x)} = \phi(\mathrm{x, k})$ for all x" by "$\mathrm{f_k(x)} = \phi(\mathrm{x, k}) + 1$ for all x".

p. 131, line 6 from the bottom, replace "$\mathfrak{C}[B]$" by "$\neg\mathfrak{C}[B]$".
— line 5 from the bottom, replace "$\mathfrak{F}[B] \to \mathfrak{C}[B]$" by "$\neg(\mathfrak{F}[B] \to \mathfrak{C}[B])$".

p. 149, last line, replace "$\neg A$" in the inference rule ($\bot_{\mathrm{C}}$) by "$(\neg A)$".

p. 161, line 7, (16.45v), replace the lower sequent "$\Gamma \Rightarrow \neg(A \to B)$" by "$\Gamma \Rightarrow \Delta, \neg(A \to B)$".

p. 182, line 15 from the bottom, delete "clause (ii) of definitiom 18.16.".
— line 17 from the bottom, delete "indexset(s)!of wffs!downward saturated".

p. 183, line 19 from the bottom, replace "$C_1 \to C_2$" by "$C_1 \wedge C_2$".

p. 189, line 18, replace

$$\text{“}\mathfrak{F}_1[\neg U \vee V, \neg(\neg U \vee \neg V)] \wedge \ldots \wedge \mathfrak{F}_{\mathrm{k}}[\neg U \vee V, \neg(\neg U \vee \neg V)]''$$

by

$$\text{“}\mathfrak{F}_1[\neg U \vee V, \neg(\neg U \vee V)] \wedge \ldots \wedge \mathfrak{F}_{\mathrm{k}}[\neg U \vee V, \neg(\neg U \vee V)]''.$$

p. 192, line 14, replace

$$\text{“}\Gamma[A] \Rightarrow \Delta[A], \Delta[A] \wedge \mathfrak{C}[A]'' \qquad \text{by} \qquad \text{“}\Gamma[A] \Rightarrow \Delta[A], \mathfrak{F}[A] \wedge \mathfrak{C}[A]''.$$

[17] With special thanks to Valerie Kerruish who detected most of the mistakes.

— lines 15 and 20, replace "$A \leftrightarrow B, \Gamma[B] \Rightarrow \Delta[B], \Delta[B] \wedge \mathfrak{C}[B]''$ by "$A \leftrightarrow B, \Gamma[B] \Rightarrow \Delta[B], \Delta[B] \wedge \mathfrak{C}[B]''$.

p. 197, line 8 from the bottom, delete "*indexvariable(s)!sentence*".

p. 198, line 2 from the bottom, replace "$\bot$-inference" by "$\bot_{\mathrm{C}}$-inference".

p. 200, the bottom:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A, \Pi \Rightarrow B\end{matrix}}{\Pi \Rightarrow A \to B}}{\Gamma, \Pi \Rightarrow B}\ .$$

instead of:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Pi \Rightarrow B\end{matrix}}{\Pi \Rightarrow A \to B}}{\Gamma, \Pi \Rightarrow B}\ .$$

p. 213, second line in top proof figure, right branch, replace "$\Theta, \Gamma[\,] \Rightarrow \Delta[\,], A, \Xi$" by "$\Theta, B, \Pi[\,] \Rightarrow \Lambda[\,], \Xi$".

p. 214, second line in top proof figure, left branch, replace "$\Pi, \Gamma[\,] \Rightarrow \Delta[\,], A, K, A \wedge B$" by "$\Pi, \Gamma[\,] \Rightarrow \Delta[\,], A, \Lambda, A \wedge B$".

p. 217, line 1, replace "$\mathrm{max(l, m) + 1 + r}$" by "$\mathrm{max(l, m) + 1 + r + 1}$".

p. 241, line 12 from the bottom, condition "(vi)": read "$\mathbf{LK}_0^0$" instead of "$\mathbf{GK}_0^0$".

p. 242, last line, read "Only the second one" instead of "Only the fourth".

p. 247, line 6, replace "dropping axioms HA13 and HA15" by "replacing axiom HA13 by $\neg\neg\bot \to \bot$ and dropping axiom HA15 completely".

p. 249, proof figure "*Re* 24.7iv", first line: read "$A \Rightarrow A \vee \neg A$" instead of "$A \Rightarrow A \Rightarrow \neg A$".

p. 250, lines 3–6 from the bottom, "(24.11i)–(24.11iv)", read "acc." instead of "max".

p. 301, line 13 from the bottom: read "contradictions" instead of "contractions".

p. 306, line 5: read “marked with the sign ♠” instead of “marked with an exclamation sign”.
— line 7 proposition 27.7: read “DSL” instead of “CDL”.

p. 307, replace proof figure in the middle of the page:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}}{A \to \neg A, A, A \Rightarrow}}{A \to \neg A, A \,\Box\, A \Rightarrow} \qquad \dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}}{\dfrac{A \to \neg A, A \to A \,\Box\, A \Rightarrow \neg A}{A \to A \,\Box\, A \Rightarrow (A \to \neg A) \to \neg A}}\;.$$

by:

$$\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A} \qquad \dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A}{\neg A, A \Rightarrow}}{A \to \neg A, A, A \Rightarrow}}{A \to \neg A, A \,\Box\, A \Rightarrow}}{\dfrac{A \to \neg A, A \to A \,\Box\, A \Rightarrow \neg A}{A \to A \,\Box\, A \Rightarrow (A \to \neg A) \to \neg A}}\;.$$

p. 309, second line: add “logic” after “dialectical”.

p. 316, first line: cancel 27.35viii; already 27.35vi;
— second line: read “$(A \diamond \neg A) \leftrightarrow \top$” instead of “$(A \diamond \neg A) \leftrightarrow \bot$”.

p. 352, l. 10 from the bottom (Corollary 30.21): read “30.21i” instead of “30.80”;
— l. 11 from the bottom (Corollary 30.21): read “30.21ii” instead of “30.81”;
— l. 12 from the bottom: read “30.20i” instead of “30.17i”.

p. 460, after the first proof figure, replace: “A new deduction is being contracted as follows” by : “A new deduction can be constructed as follows”.

p. 466, l. 12 from the bottom, Definition 41.6: swap (1) and (2).

p. 468, last three lines: replace Definitions 41.14 by the following:

DEFINITIONS 41.14 (1) $uni[\mathfrak{F}] :\equiv \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \square \mathfrak{F}[z_2] \rightarrow z_1 = z_2)$.
(2) $\iota x\, \mathfrak{F}[x] :\equiv \lambda x \bigwedge y\, (uni[\mathfrak{F}] \square \mathfrak{F}[y] \rightarrow x \in y)$.

p. 469, replace PROPOSITION 41.15 by the following:
PROPOSITION 41.15. *Sequents according to the following schemata are* $\mathbf{LX}_\lambda^-$*-deducible for* $\mathbf{X} \in \{\mathbf{K}, \mathbf{J}, \mathbf{P}, \mathbf{D}\}$.

(41.15i) $s \in \iota x\, \mathfrak{F}[x], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \square \mathfrak{F}[z_2] \rightarrow z_1 = z_2), \mathfrak{F}[t] \Rightarrow s \in t$

(41.15ii) $s \in t, \mathfrak{F}[t] \Rightarrow s \in \iota x\, \mathfrak{F}[x]$

(41.15iii) $\bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \square \mathfrak{F}[z_2] \rightarrow z_1 = z_2), \mathfrak{F}[t] \Rightarrow \iota x\, \mathfrak{F}[x] = t$

Proof. *Re* 41.15i.

$$\begin{array}{c} uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow uni[\mathfrak{F}] \square \mathfrak{F}[t] \qquad s \in t \Rightarrow s \in t \\ \hline uni[\mathfrak{F}] \square \mathfrak{F}[t] \rightarrow s \in t, uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow s \in t \\ \hline \bigwedge y\, (uni[\mathfrak{F}] \square \mathfrak{F}[y] \rightarrow s \in y), uni[\mathfrak{F}], \mathfrak{F}[t] \Rightarrow s \in t \\ \hline s \in \iota x\, \mathfrak{F}[x], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \square \mathfrak{F}[z_2] \rightarrow z_1 = z_2), \mathfrak{F}[t] \Rightarrow s \in t \end{array}\,.$$

Re 41.15ii.

$$\begin{array}{c} \mathfrak{F}[t], \mathfrak{F}[b] \Rightarrow \mathfrak{F}[t] \square \mathfrak{F}[b] \qquad b = t, s \in t \Rightarrow s \in b \\ \hline s \in t, \mathfrak{F}[t] \square \mathfrak{F}[b] \rightarrow b = t, \mathfrak{F}[t], \mathfrak{F}[b] \Rightarrow s \in b \\ \hline\hline s \in t, \mathfrak{F}[t], \bigwedge z_1 \bigwedge z_2\, (\mathfrak{F}[z_1] \square \mathfrak{F}[z_2] \rightarrow z_1 = z_2), \mathfrak{F}[b] \Rightarrow s \in b \\ \hline s \in t, \mathfrak{F}[t], uni[\mathfrak{F}], \mathfrak{F}[b] \Rightarrow s \in b \\ \hline\hline s \in t, \mathfrak{F}[t], uni[\mathfrak{F}] \square \mathfrak{F}[b] \Rightarrow s \in b \\ \hline s \in t, \mathfrak{F}[t] \Rightarrow uni[\mathfrak{F}] \square \mathfrak{F}[b] \rightarrow s \in b \\ \hline s \in t, \mathfrak{F}[t] \Rightarrow \bigwedge y\, (uni[\mathfrak{F}] \square \mathfrak{F}[y] \rightarrow s \in y) \\ \hline s \in t, \mathfrak{F}[t] \Rightarrow s \in \iota x\, \mathfrak{F}[x] \end{array}\,.$$

Re 41.15iii. This is a fairly straightforward combination of 41.15i and ii. Left to the reader. QED

p. 484, l. 9 from the bottom "41.57iv": read "$t \in \mathbf{T}, s' = t', r \in s \Rightarrow r \in t$" instead of "$t \in \mathbf{T}, s' = t', s \in r \Rightarrow s \in t$".

p. 491, "*Re* 41.72iii": involves a cut which doesn't make it suitable for $\mathbf{LP}_\lambda$.

p. 492, proof figure "*Re* 41.74ii", replace "$\langle \langle s, 0' \rangle, \mathfrak{g}(s, 0, \mathfrak{f}(s)) \rangle \in \mathfrak{h}$" by

"$\langle\langle s,0\rangle,\mathfrak{f}(s))\rangle\in\mathfrak{h}$"

p. 495, proof figure "*Re* 41.78i", second line: add "$t\in\mathbf{N},\langle\langle s,t\rangle,r\rangle$" before "$\in\mathfrak{h}$,"

— proof figure "*Re* 41.79i", replace lower sequent by

$$s\in\mathbf{N}\Rightarrow\langle\langle s,0\rangle,\mathfrak{h}(s,0)\rangle\in\mathfrak{h}\,.$$

p. 499, l. 10 from the bottom (in Remark 42.3): replace

$$\lambda x\centerdot t\equiv\{z:\bigvee x\bigvee y\,(z=\langle x,y\rangle)\,\square\,y=t\}\,,$$

by

$$\lambda x\centerdot t\equiv\{z:\bigvee x\bigvee y\,(z=\langle x,y\rangle\,\square\,y=t)\}\,,$$

— l. 3 from the bottom: replace **D** by **P**.

p. 502, l. 11 (Definition 42.11. clause (6)) add: "with z being the first variable $\notin FV(AB)$".

p. 564, l. 6 from the bottom: "obtained from them" instead of "obtained form them".

p. 570, l. 6: read "$\mathbf{LX}_1^{\mathrm{Q}}$-admissible" instead of "$\mathbf{HX}_1^{\mathrm{Q}}$-admissible".

p. 572, l. 2: "the formal principles" instead of "the formal principle"

— l. 4: "The remainder of this section" instead of "The remainder this section"

p. 574, l. 14 and 15: "$\bigwedge x\,\mathfrak{F}[x]$" instead of "$\bigwedge x\,[x]$"

p. 586, l. 4 (proof figure, second line): read "$b<c,c<a',b<a\rightarrow\neg\mathfrak{F}[b]$" instead of "$\neg\mathfrak{F}[b]$)"

— last line: replace

$$\neg\neg\bigvee x\,\mathfrak{F}[x]\Rightarrow\bigvee y\,(\mathfrak{F}[y]\wedge\bigwedge z\,(z<y\rightarrow\neg\mathfrak{F}[z]))\,.$$

by

$$\neg\neg(\bigvee x\,\mathfrak{F}[x]\rightarrow\bigvee y\,(\mathfrak{F}[y]\wedge\bigwedge z\,(z<y\rightarrow\neg\mathfrak{F}[z])))\,.$$

p. 605, l. 18 from the bottom (Definition 48.4 (2)): replace "*fof*(r,s)" by "*fof*(r)".

p. 607, l. 6 from the bottom (main text): replace "free variables.footnote" by "free variables." and read the sentence beginning with "Primitive recursive functions" and ending with "variables in PRA." as a footnote.

p. 621, l. 4 (REMARK 48.61): read

$$\bigwedge x\,(\mathfrak{F}[x] \vee \neg\mathfrak{F}[x]) \Rightarrow \bigvee e \bigwedge x\,((\mathfrak{F}[x] \leftrightarrow \phi_e(x)=1) \wedge (\phi_e(x)=0 \vee \phi_e(x)=1)),$$

instead of

$$\bigwedge x\,(\mathfrak{F}[x] \vee \mathfrak{F}[x]) \Rightarrow \bigvee e \bigwedge x\,((\mathfrak{F} \leftrightarrow \phi_e(x) = 1) \wedge (\phi_e(x) = 0 \vee \phi_e(x) = 1)).$$

p. 739, l. 19 (QUOTATION 57.27): read “in very few cases or none” instead of “in very few cases or non”.

p. 894, l. 4 from the bottom (footnote 2): replace “Wang [1986]” by “Wang [1987]”.

p. 1011, l. 8: read “from” instead of “form”.

p. 1017, l. 14 from the bottom: read “it follows” instead of “if follows”.

p. 1026, l. 13 from the bottom: read “This sounds like” instead of “This sound like”.

p. 1030, l. 16: read “from the value” instead of “form the value”.

p. 1081, l. 6, QUOTATION 76.16. (1), add: “Weyl” before “[1921]”.

p. 1087, l. 23 from the bottom, QUOTATION 77.8. (1), new paragraph after “meaningless.” and before “In all contexts ...”.

p. 1087, l. 14 from the bottom, QUOTATION 77.8. (1), add: “put” before “numerals for the variables in such a way ...”.

p. 1088, l. 22, QUOTATION 77.9. replace “ist” by “is”: “correlate of a subclass is that subclass itself”.

p. 109, after l. 24 (QUOTATION 78.12) add line: “expresses a true proposition with respect to every one of *these* models, we”

p. 1099, l. 11 from the bottom: replace “67.20” by “78.17”.

p. 1102, l. 13: replace “Wang [1986]” by “Wang [1987]”.

p. 1104, l. 4 from the bottom: replace “Wang [1986]” by “Wang [1987]”.

p. 1106, l. 3: replace “Wang [1986]” by “Wang [1987]”.

p. 1108, l. 16: replace “form” by “from” in QUOTATION 79.10.

p. 1109, l. 1: replace “Wang [1986]” by “Wang [1987]”.

p. 1126, l. 19: replace "Wang [1986]" by "Wang [1987]".

p. 1133, l. 4: replace "*Brouwerian*" by "*Brouwerians*".

p. 1158, l. 3 from the bottom: replace "Wang [1986]" by "Wang [1987]".

p. 1161, l. 16: replace "Wang [1986]" by "Wang [1987]".

p. 1169, l. 4: read "share no fixed point" instead of "share not fixed point".

p. 1185, l. 16 from the bottom (QUOTATION 85.10 (4)): read "$\forall x_1 \ldots \forall x_n t(x_1, \ldots, x_n = \overline{o})$" instead of "$\forall x_1 \ldots \forall x_n(t = \overline{o})$".

p. 1207, l. 5: replace "Wang [1986]" by "Wang [1987]".

p. 1300, l. 1: read "how many angels" instead of "how man angels".

p. 1303, l. 2: read "*d'être* for the" instead of "*d'être* n for the".
— l. 14 from the bottom: read "Gödel" instead of "G"odel".

p. 1307, l. 1: read "amenable" instead of "amendable".

p. 1368, l. 7 from the bottom, "(3) Girard [1995]": replace "p. 28" by "p. 171";
— l. 4 from the bottom: read "work" instead of "word";
— last line (of text): read "[1982]" instead of "[1974]".

p. 1387, l. 8 from the bottom (footnote 11): replace "Wang [1986]" by "Wang [1987]".

p. 1412, l. 4: replace "and Wandschneider [1984]" by "Kesselring [1984], and Wandschneider [1991]".

p. 1546, l. 13 from the bottom: replace "ἀριϑμίζει" by "ἀριϑμητίζει".

p. 1421, l. 9 from the bottom (disregarding footnotes): replace "ancient means "never"." by "ancient means "ever", "once"."

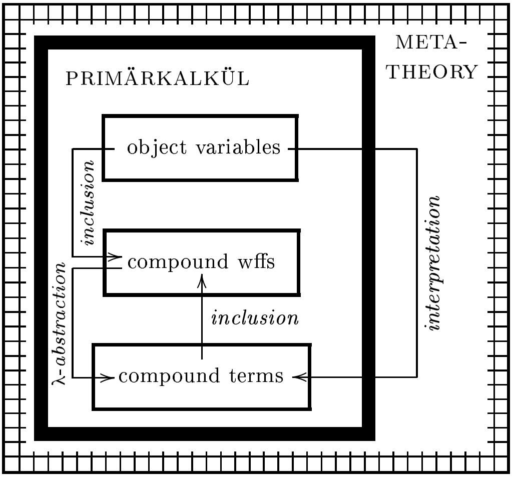
p. 1557, l. 6 from the bottom: delete "Take, *e.g.*, *tertium non datur* for negated wffs, $\neg A \vee \neg\neg A$; this is perfectly provable in intuitionistic logic". This is utter nonsense and I have no idea what was going on in my mind when I wrote it. Perhaps I was thinking of 'double negation', $\neg\neg A \rightarrow A$, which holds intuitionistically for negated wffs: $\neg\neg\neg A \rightarrow \neg A$. This is what it can be replaced by: "Take, *e.g.*, the double negation of *tertium non datur*, $\neg\neg(A \vee \neg A)$; this is perfectly provable in intuitionistic logic".

p. 1571, l. 2, replace the topmost proof figure by the following one:

$$\dfrac{\dfrac{\mathrm{R}\in\mathrm{R} \Rightarrow \mathrm{R}\in\mathrm{R}}{\Rightarrow \mathrm{R}\in\mathrm{R}, \mathrm{R}\notin\mathrm{R}} \qquad \dfrac{\dfrac{\dfrac{\mathrm{R}\in\mathrm{R} \Rightarrow \mathrm{R}\in\mathrm{R}}{\mathrm{R}\notin\mathrm{R}, \mathrm{R}\in\mathrm{R} \Rightarrow}}{\mathrm{R}\in\mathrm{R}, \mathrm{R}\in\mathrm{R} \Rightarrow}\in}{\mathrm{R}\in\mathrm{R} \Rightarrow}\spadesuit}{\Rightarrow \mathrm{R}\notin\mathrm{R}}\clubsuit$$

and (on the same page), in second proof figure, last line, read "$\mathrm{R}\notin\mathrm{R} \Rightarrow$" instead of "$\mathrm{R}\in\mathrm{R} \Rightarrow$".

p. 1601: replace diagram 116.10 by the following one:



p. 1621, l. 9 from the bottom: replace "$\bigwedge y$" by "$\bigwedge Y$".
p. 1630, l. 18: insert ")" before "$\Rightarrow$"; *i.e.*, replace "$(\mathfrak{F}[t_2] \vee \neg\mathfrak{F}[t_2] \Rightarrow$" by "$(\mathfrak{F}[t_2] \vee \neg\mathfrak{F}[t_2]) \Rightarrow$".
p. 1669: in the proof figure "*Re* 123.13ii", sixth line: replace "$\lambda\top \sqsubseteq \bigwedge x\,(\mathfrak{F}[x] \to \widetilde{b})$" by $\lambda\top \sqsubseteq \lambda \bigwedge x\,(\mathfrak{F}[x] \to \widetilde{b})$".
In remark 123.14, second line, replace "that" by "than".

p. 1670: in the proof figure "*Re* 123.18i" replace "$\perp$" by "$\top$" throughout. In the proof figure "*Re* 123.18ii" replace the second line "$A \to \widetilde{a} \Rightarrow B \to \widetilde{a}$" by "$B \to \widetilde{a} \Rightarrow A \to \widetilde{a}$" and the third $C \to (A \to \widetilde{a}) \Rightarrow C \to (B \to \widetilde{a})$" by "$C \to (B \to \widetilde{a}) \Rightarrow C \to (A \to \widetilde{a})$".

In the proof figure "*Re* 123.18iii" replace the second line "$A \to (C \to \widetilde{a}) \Rightarrow B \to (C \to \widetilde{a})$" by "$B \to (C \to \widetilde{a}) \Rightarrow A \to (C \to \widetilde{a})$".

p. 1705, (125.12iii): replace

$$s \ddot{=} t \Rightarrow \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} s) \ddot{\subseteq} \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} t))$$

by

$$s \ddot{\subseteq} t \Rightarrow \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} s) \sqsubseteq \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} t))$$

— *Re* 125.12iii: Replace proof figure by:

$$\dfrac{s \ddot{\subseteq} t, \lambda y\, (y \ddot{=} a) \ddot{\subseteq} s \Rightarrow \lambda y\, (y \ddot{=} a) \ddot{\subseteq} t}{s \ddot{\subseteq} t \Rightarrow \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} s) \sqsubseteq \lambda x\, (\lambda y\, (y \ddot{=} x) \ddot{\subseteq} t)}\ .$$

p. 1706, l. 2, proof of proposition 125.15, replace the whole thing by the following:

Proof. By means of contraction it is possible to prove a contradiction along similar lines as for 141.11 in the appendix A1, only with somewhat more rudimentary notions due to the lack of 41.2 (1.2) and (2.3). Take R to be defined as $\lambda x\, (\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)$ and consider the following deductions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \qquad \dfrac{\Rightarrow a \ddot{=} a \qquad \dfrac{\ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\subseteq} a}{\ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{\ddot{\{}a\ddot{\}} \ddot{\subseteq} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{\lambda x\, (\ddot{\{}x\ddot{\}} \ddot{\subseteq} a) \sqsubseteq \lambda x\, (\ddot{\{}x\ddot{\}} \ddot{\subseteq} \mathrm{R}), \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{a \ddot{\subseteq} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}\ 125.14}{a \ddot{=} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}}{a \ddot{=} \mathrm{R}, \ddot{\{}a\ddot{\}} \ddot{\subseteq} a \Rightarrow}\ \spadesuit}{a \ddot{=} \mathrm{R} \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a}}{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \ddot{\subseteq} \lambda x\, (\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)}\ 125.7\mathrm{i}\,,$$

and also:

$$\dfrac{a \ddot{=} \mathrm{R} \Rightarrow \ddot{\{}a\ddot{\}} \ddot{\not\subseteq} a}{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \sqsubseteq \lambda x\, (\ddot{\{}x\ddot{\}} \ddot{\not\subseteq} x)}\ .$$

Continue as follows:

$$\dfrac{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \mathrel{\ddot{=}} \ddot{\{}\mathrm{R}\ddot{\}} \qquad \dfrac{\Rightarrow \ddot{\{}\mathrm{R}\ddot{\}} \mathrel{\ddot{\subseteq}} \mathrm{R}}{\ddot{\{}\mathrm{R}\ddot{\}} \mathrel{\ddot{\not\subseteq}} \mathrm{R} \Rightarrow}}{\ddot{\{}\mathrm{R}\ddot{\}} \sqsubseteq \mathrm{R} \Rightarrow}\ .$$

QED

p. 1729, l. 4 from the bottom: insert round bracket after $\|^{\mathrm{DL}}$:
$\neg\neg(s \in \|\lambda x\, \mathfrak{C}[x]\|^{\mathrm{DL}}) \Rightarrow s \in \|\lambda x\, \mathfrak{C}[x]\|^{\mathrm{DL}}$.

p. 1737, first line: replace "Employ 126.64ii" by "Employ 126.63i".

p. 1759, l. 5, (128.27ii), replace h by f: $r_1 = s_1, r_2 = s_2, f[\![r_1, r_2]\!] = t \Rightarrow f[\![s_1, s_2]\!] = t$.

p. 1763, l. 15 from the bottom "128.34ii": read "$\Rightarrow\ 0 \in \mathbf{T}$" instead of "$t \in \mathbf{T}, s \in r, r \in t \Rightarrow s \in t$".

p. 1809, l. 16 from the bottom (proof of lemma 132.13, third last line): read "By proposition 126.35" instead of "By proposition 131.22".

p. 1818, l. 7 from the bottom (PROPOSITION 133.8): read "$\mathcal{D}$ has the left rank 1 " instead of "$\mathcal{D}$ has the rank 1".
— l. 5 from the bottom: read "If the left rank were" instead of "If the rank were".

p. 1823, l. 6 from the bottom, replace

$$\mathit{\Gamma}, \breve{\gamma}[\textstyle\bigwedge z\,(z \in b \rightarrow z^I \in b)] \Rightarrow s \in b$$

by

$$\mathit{\Gamma}, \breve{\gamma}[\textstyle\bigwedge z\,(z \in b \rightarrow z^I \in b)] \Rightarrow s \in s\,.$$

— Second last line, replace

$$\mathit{\Gamma}, [\textstyle\bigwedge z\,(z \in b \rightarrow z^I \in b)]^{\cdot \mathrm{n}} \Rightarrow s \in b$$

by

$$\mathit{\Gamma}, [\textstyle\bigwedge z\,(z \in b \rightarrow z^I \in b)]^{\cdot \mathrm{n}} \Rightarrow s \in c\,.$$

p. 1824, second line in the proof of theorem 135.15, replace "$\delta_1 \geq 0$" by "$\delta_1 > 0$".

p. 1825, DEFINITION 134.1., replace

$$\breve{\mathbf{\Pi}}^{\circ} :\equiv \lambda x\,(x \in \mathbf{Z} \,\square\, \textstyle\bigwedge y\,(I \in y \,\square\, [\bigwedge z\,(z \in y \rightarrow z^I \in y)/x] \rightarrow x \in y))$$

by

$$\breve{\mathbf{\Pi}}^{\circ} :\equiv \lambda x\,(x\in\mathbf{Z} \,\square\, \bigwedge y\,([I\in y \wedge \bigwedge z\,(z\in y \to z^{I}\in y)/x] \to x\in y))\,.$$

p. 1830, first line (134.9iv), replace “131.18ii” by “131.18i”.

p. 1832, second last line (134.16iii), replace

$$\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda} \cup \{\Box(\Box\bot \to \bot) \to \Box\bot\} \vdash \bot\Box(\Box\bot \to \bot) \to \Box\bot$$

by

$$\mathbf{L^{i}D}^{\mathrm{Z}}_{\lambda} \cup \{\Box(\Box\bot \to \bot) \to \Box\bot\} \vdash \bot\,.$$

p. 1834, replace last proof figure on that page (“*Re* 134.22i.”) by the following one:

$$\dfrac{\Box\Gamma \Rightarrow \Box C \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow A}{C \Rightarrow A}}{\Rightarrow C \to A}}{\Rightarrow \Box(C \to A)} \qquad \Box(C \to A), \Box C \Rightarrow \Box A}{\Box C \Rightarrow \Box A}\,\clubsuit}{\Box\Gamma \Rightarrow \Box A}\,\clubsuit\,.$$

p. 1842, third line, replace

$$[B/s], [A/I \sqcap I] \to [B/I], [A/s], [A/I] \Rightarrow [B/s \sqcap I]$$

by

$$[B/s], [A/I \sqcap I] \to [B/I], [A/I], [A/I] \Rightarrow [B/s \sqcap I]$$

p. 1848, l. 12 from the bottom, first line in the deduction *re* 135.20vii, replace “$A_2 \Rightarrow A_1$” by “$A_2, \Gamma \Rightarrow A_1$” and cancel “$B_1, A_2, A_2, \Gamma, \Pi \Rightarrow B_2$” completley.

— l. 11 from the bottom, second line of that deduction, replace

“$a\in\breve{\mathbf{\Pi}}, [A_2/a] \Rightarrow [A_1/a]$” by “$a\in\breve{\mathbf{\Pi}}, [A_2/a], \Gamma \Rightarrow [A_1/a]$”,

and

“$B_1, A_2, A_2, \Gamma, \Pi \Rightarrow B_2$” by “$B_1, A_2, A_2, \Pi \Rightarrow B_2$”.

p. 1886, third line (137.8i), as well as line 10 and 11 (in proof figure), replace “$\Rightarrow zhf[s_1, t_1, r_1]$” by “$\Rightarrow zhf[s_2, t_2, r_2]$”;

— fourth line (137.8ii), replace “$\Rightarrow shg[s_1, t_1, r_1]$” by “$\Rightarrow shg[s_2, t_2, r_2]$”;

p. 1901, sixth line from the bottom, replace “$\to y\in T$” by “$\to y\in \mathcal{T}$”.

— l. 3 from the bottom, replace "$\to \mathcal{T}$" by "$\to y \in \mathcal{T}$".

p. 1903, first line, replace the quantifier $\bigvee$ in the wff

$$\neg \bigvee x \, (ded_{\mathbf{L^iD}^{\mathrm{z}}_{\lambda}}(x, \ulcorner \mathrm{G} \urcorner) = 0) \leftrightarrow \mathrm{G}$$

by the quantifier $\bigvee^{\circ}$:

$$\neg \bigvee^{\circ} x \, (ded_{\mathbf{L^iD}^{\mathrm{z}}_{\lambda}}(x, \ulcorner \mathrm{G} \urcorner) = 0) \leftrightarrow \mathrm{G}\,.$$

p. 1923, second proof figure, replace

$$\dfrac{\Rightarrow \mathrm{C}_A \in \mathrm{C}_A \qquad \dfrac{\dfrac{\mathrm{C}_A \in \mathrm{C}_A \Rightarrow A}{\Rightarrow \mathrm{C}_A \in \mathrm{C}_A \to A}}{\mathrm{C}_A \in \mathrm{C}_A \Rightarrow A}}{\Rightarrow A}\ \clubsuit\,.$$

by

$$\dfrac{\dfrac{\dfrac{\mathrm{C}_A \in \mathrm{C}_A \Rightarrow A}{\Rightarrow \mathrm{C}_A \in \mathrm{C}_A \to A}}{\Rightarrow \mathrm{C}_A \in \mathrm{C}_A} \qquad \mathrm{C}_A \in \mathrm{C}_A \Rightarrow A}{\Rightarrow A}\ \clubsuit\,.$$

p. 1925, *Re* 141.iii and iv, third proof figure from the top, replace "$\Leftrightarrow \mathrm{K} \in \lambda x \, (\mathsf{C} x \notin x)$" by "$\Leftrightarrow \mathrm{K} \in \lambda x \, (\mathsf{C} x \in x)$"

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