Naive abstraction and truth

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Prologue: state of the art

In reconsidering the so-called naive principles for sets as well as for truth, typically one can follow two routes:

- naive abstraction is suitably restricted (e.g. with positivity conditions), but it is projected into classical or intuitionistic logic;
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Comprehension and extensionality

Non-uniform naive comprehension CA: for A arbitrary, $y \notin FV(A)$

$$(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow A(u, x))$$

CA states that there exists a binary relation E on the universe U which is universal for U-subsets, and this is impossible due to Cantor's theorem, as one could define a surjection of U onto its power set.

Possible way out: syntactical restrictions reflecting topological ideas ...

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If $=$ is primitive, Ext has the usual form, i.e. equiextensional sets are equal

$$x =_e y \to x = y$$

where $x =_e y$ is $(\forall z)(z \in x \leftrightarrow z \in y)$. Else, if $=$ is not primitive, Ext means

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Generalized positive formulas GPF: the smallest class containing atoms $t \in s$, $t = s$, closed under $\wedge$, $\vee$, $\forall$, $\exists$ and also bounded qtfs $\forall x(x \in y \rightarrow \ldots)$ and universal qtfs restricted to definable classes $\forall x(C(x) \rightarrow \ldots))$.

Theorem (Malitz 1976, Weydert 1988, Forti-Hinnion 1989)

CA for GPF-formulas (hence Pos(=)-CA+Ext) is consistent.

Proof: the so-called hyperuniverses (Forti-Honsell 1994), topological models. **Non-uniformity of CA essential**! . . .
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Abstraction

Uniform comprehension, i.e. abstraction $\mathcal{F}$-AP, the uniform CA, or abstraction principle:

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Let Pos($\in, \neq, =$) be the class of positive fmlas generated from positive $\in$-atoms and *positive and negative* $=$-atoms.

Theorem (... Gilmore...)

Pos($\in, \neq, =$)-AP is consistent

Model: the universe is given by terms with literal identity, while the interpretation of $\in$ is inductively generated (exploit positivity and Tarski-Knaster). Much more is true (possibly enlarge the language with dual membership...).

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Pos($\in, =$))*-AP is inconsistent with Ext*

Theorem (CM 99)

$QF^{+}(\in, =)$*-AP is inconsistent with*

- *the power set axiom;*
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Ordered pairing $x, y \mapsto \langle x, y \rangle$ can be defined as usual...

Theorem ($Pos(=, \in)$-AP)

If $fa := \{x \mid \langle x, a \rangle \in f\}$, then there is a term I_f with $FV(I_f) = FV(f)$ such that

$$\Rightarrow I_f =_e fI_f$$

If $t(x)$ is an arbitrary term, there exists I such that

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For the proof, choose

$$D_f = \{z \mid \exists x \exists g (z = \langle x, g \rangle \otimes x \in f(gg))\} \quad (1)$$

$$I_f = D_f D_f \quad (2)$$

On the other hand, if $=$ is omitted:

Theorem (Hinnion, Libert 2003)

Pos($\in$)-AP is consistent with Ext.

Construction: inductive generation of $\in$ on the term model; then show that equiextensionality is a congruence in the fixed point model also with respect to abstraction terms!

Remark. Libert 2007: domain-theoretic construction. Untyped lambda calculus extended with Fregean notions once beta conversion is restricted to positive expressions (i.e. $\neg$, $=$ and $\rightarrow$ are omitted).

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Foundational Applications?

No chance to regain a sort of Frege-Russell paradise. But untyped positive AP is useful for designing a predicative universe on the top of an underlying rich basis (arithmetic, models of combinatory logic).

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Formal language

$\mathcal{L}_S$ is the elementary set theoretic language, which comprises

1. the binary predicate symbol $\in$;
2. the logical symbols $\rightarrow$, $\wedge$, $\vee$, $\otimes$, $+$, $\exists$, $\forall$, the propositional constants $\bot$, $\top$.
3. the abstraction operator $\{-|-\}$;
4. individual variables $(x, y, z, \ldots)$.

$\bot$ is the absurd proposition; $\top$ is the true proposition;$\rightarrow$ stands for a substructural implication: $A \rightarrow B$ roughly means that B follows from A via a deduction which uses the assumption A at most once;$\wedge$, $\vee$ ($\otimes$, $+$) denote the so-called additive (multiplicative) conjunction and disjunction of contractionless logic;finally, $\exists$, $\forall$ stand for the usual quantifiers.

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General abstraction operator

Generalized class terms:

if φ is a formula, and $t(\vec{x})$ is a term whose free variables occur in the list $\vec{x}$, then $\{t(\vec{x}) \mid \varphi\}$ is a term where $FV(\{t(\vec{x})|\varphi\}) = FV(\varphi) - \{\vec{x}\}$, $FV(E)$ is the set of free variables occurring in the expression E)

NB: if $t(\vec{x}) := x$, we get usual abstraction. If $t(\vec{x})$ is **injective**, we can derive RAP from AP by choosing as usual:

$$\{t(\vec{u})|A(\vec{u},\vec{w})\} = \{v|(\exists\vec{x})(v = t(\vec{x}) \otimes A(\vec{x},\vec{w}))\}$$

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Extending Grishin

GSR is Grishin's system with RAP, the schema

$$(\forall\vec{v})(\forall\vec{x})(t(\vec{x}) \in \{t(\vec{u})|A(\vec{u},\vec{v})\} \leftrightarrow A(\vec{x},\vec{v})).$$

(NB: a *new* binding operator)

Theorem

Cut rule is admissible in GSR and hence GSR is consistent.

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Fixed points again

The fixed point construction need not use standard logic: contraction free is enough !

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If $fa := \{x \mid \langle x, a\rangle \in f\}$, then there is a term I_f with $FV(I_f) = f$ such that, provably in GSR:

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Clearly, using RAP, we can choose (no need of existential quantifiers):

$$D_f = \{\langle x, g\rangle \mid x \in f(gg)\}$$

NB: why do we restrict logic and yet maintain unrestricted term formation? See the non-linear feature of the term D_f.

Application I: non-extensionality

Extensionality can now be easily refuted, e.g. for the empty set $\emptyset$

Proof's hint: choose g such that

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Application II: undecidability

Representing combinatory logic CL

The relation "$t = s$ is equationally provable in combinatory logic", i.e. formally CL $\vdash t = s$ is the smallest equivalence relation on terms, generated by the initial conditions $Kab = a$ and $Sabc = ac(bc)$, and closed under the inferences:

$$a = b \quad \Rightarrow \quad ac = bc$$
$$a = b \quad \Rightarrow \quad ca = cb$$

CL *is essentially undecidable*.

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CL *is essentially undecidable*.

Let TER_{CL} be the set of CL-terms and let TER_{GS} be the set of GS-terms. Then:

Theorem

There exist:

(i) *a translation* $\widehat{t} \ : \ TER_{CL} \ \mapsto \ TER_{GS}$

(ii) *a closed term* $\mathcal{E}$ *in GS such that*

$$CL \vdash t = s \Leftrightarrow GS \vdash \Rightarrow \langle \widehat{t}, \widehat{s} \rangle \in \mathcal{E}$$

Hence GS is undecidable

As to the main steps the proof, by fixed point we can simulate the syntax of CL, the definition of CL-derivability and natural numbers. For instance, if we define

$$\begin{aligned} \overline{0} &:= \emptyset; \\ t+1 &:= \{t\}; \\ \overline{n+1} &:= \overline{n}+1, \end{aligned}$$

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By application of the contraction free nature of the calculus (e.g., restricted invertibility of the $\exists$-introduction rule to the right, given that the antecedent is empty), it is not difficult to check:

1. if GS $\vdash\ \Rightarrow\ t = s$, then $t \equiv s$ ("the literal identity property");
2. if GS $\vdash\ \Rightarrow\ t \in \omega$, then for some natural number n, $t \equiv \overline{n}$ ("the ω–evaluation property").

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1. Consistency of AP with Grishin's logic + Dummett's law?
2. Consistency of AP with Grishin's logic + Dummett's law + $\wedge$-commutativity?

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Hypersequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

Standard interpretation

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where

- $\Pi\Gamma_i = A_1 \otimes \ldots \otimes A_k$, if $\Gamma_i \neq \emptyset$; else $\Gamma_i = \top$;
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Hypersequent Calculus IMTL∀

Grishin+Linearity + Quantifiers.
Some crucial inferences

- External structural rules, e.g.

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}$$

- Communication rule Pottinger, Avron):

$$\frac{G \mid \Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \qquad G \mid \Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}$$

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Other examples:

- Cut:

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IMTL∀ derives the law of constant domains:

$$(\forall x)(A \vee B(x)) \rightarrow A \vee (\forall x)B(x)$$

Let GSRL be Grishin's system with underlying IMTL∀-logic and the comprehension schema RAP.

Conjecture

GSRL enjoys cut elimination.

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Many-valued logics?

Summary:

1. Three-valued is not enough (Mow-Shaw-Kwei 1954: can reproduce a Curry-like paradox);
2. infinitely valued is enough; partial solutions (Chang, Fenstad);
3. the solution?
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Frege Ł-theories and structures

Language: includes basic statements: $t = s$, Ts (s is true)
TŁ∀ is a theory of self-referential truth based on combinatory logic, (the finite fragment of) Ł∀ and the fixed point axiom embodying the natural closure conditions on the truth predicate:

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TŁ∀ proves:

$$\begin{aligned} T[x = y] &\leftrightarrow x = y \\ T(x \dot{\rightarrow} y) &\leftrightarrow Tx \rightarrow Ty \\ T(\dot{\neg} x) &\leftrightarrow \neg Tx \\ T(\dot{\forall} f) &\leftrightarrow (\forall x) T(fx) \end{aligned}$$

Moreover, if A is arbitrary:

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NB:formulas are encoded via terms of the underlying combinatory logic, i.e. it is possible to define a map from formulas to terms such that

$$A \mapsto [A]$$

and the free variables of A and $[A]$ coincide. Abstraction can be defined

$$\{x \mid A\} := \lambda x.[A]$$

Comparison: classical Frege structure

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\begin{array}{rcl}
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Semantics

- A countable structure $\mathcal{M}$ with domain M, which has the form:

$$\langle M, App_M, K_M, S_M, =_M \rangle$$

  where: $App_M : M \times M \rightarrow M$, $K_M, S_M \in M$, $=_M$ is crisp (its characteristic function is boolean), and $\mathcal{M}$ defines a realization of the language of TŁ$\forall$, except the truth predicate T;
- If t is an arbitrary closed term of $\mathcal{L}_{cat}(\mathcal{M})$, $||t||_M \in M$= the standard classical value is inductively defined as usual s.t. $||\{x|A\}|| = ||\lambda x[A]||$ (M omitted).

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Definition

If A is an arbitrary closed formula of $\mathcal{L}_{cat}(\mathcal{M})$, EQ is the characteristic function of crisp equality on M, and $\varphi \in [0,1]^\omega$

$$
\begin{aligned}
\|t = s\|^\varphi &:= EQ(\|t\|_M, \|s\|_M) \\
\|Tt\|^\varphi &:= \varphi(\|t\|) \\
\|A \to B\|^\varphi &:= \|A\|^\varphi \Rightarrow_L \|B\|^\varphi \\
\|\neg A\|^\varphi &:= \neg_L \|A\|^\varphi \\
\|\forall v_i A\|^\varphi &:= \inf\{\|A(a)\|^\varphi \mid a \in M\} \\
\|\exists v_i A\|^\varphi &:= \sup\{\|A(a)\|^\varphi \mid a \in M\}
\end{aligned}
$$

In the previous definition we have of course used the Łukasiewicz logical functions:

1. $a \Rightarrow_L b = \min\{1, 1 - a + b\}$;
2. $\neg_L a = 1 - a$

It can be verified that

- $a \otimes b = \max\{0, a + b - 1\}$;
- $a + b = \max\{1, a + b\}$
- $a \wedge b = \min\{a, b\}$;
- $a \vee b = \max\{a, b\}$

Definition

Every sentence A defines a function

$$F_A : [0,1]^\omega \rightarrow [0,1], \quad (3)$$

such that, if $\varphi \in [0,1]^\omega$, then $F_A(\varphi) = \|A\|^\varphi$. If $A(v)$ is a formula with *the free variable shown only*, then we define a function

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Problem

Find $\varphi \in [0,1]^\omega$ such that, whenever $\mathsf{T}Ł\forall \vdash A(a_0, \ldots, a_k)$, then $\|A(a_0, \ldots, a_k)\|^\varphi = 1$, for every sequence $a_0, \ldots, a_k$ of elements of M (k being such that $FV(A) \subseteq \{x_0, \ldots, v_k\}$).

Continuity for the truth operator? Partial result.

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Lemma

If $A(v)$ is a quantifier-free formula with at most one free variable, then the associated operator $F_A : [0,1]^\omega \rightarrow [0,1]^\omega$ is continuous (with respect to the product topology)

Hence:

Lemma ("Tychonoff-Schauder...")

Every continuous function F from $[0,1]^\omega$ into itself has a fixed point, i.e. there exists φ such that $F(\varphi) = \varphi$.

Let qf-TŁ∀ be the Frege theory restricted to quantifier-free conditions.

Theorem

There exists $\varphi \in [0,1]^\omega$, such that if qf-TŁ∀$\vdash A(v_0, \ldots, v_k)$ and $FV(A) \subseteq \{v_0, \ldots, v_k\}$, then

$$\|A(a_0, \ldots a_k\|^\varphi = 1$$

for every $a_0, \ldots a_k$ of M.

Proof

Apply the fixed point lemma to the function $F_{\mathcal{Q}}$ defined by the truth defining operator $\mathcal{Q}$ for quantifier-free conditions. Then there exists φ of $[0,1]^{\omega}$ such that $F_{\mathcal{Q}}(\varphi) = \varphi$; hence, for every $a \in \omega$,

$$\|\mathcal{Q}(a, T)\|^{\varphi} = \|T(a)\|^{\varphi} \tag{5}$$

which implies $\|(\forall x)(\mathcal{Q}(x, T) \leftrightarrow T(x)\|^{\varphi} = 1$. $\square$

Corollary

The quantifier free abstraction schema is consistent in the logic Ł∀.

This strengthens Skolem's original proof (for the non-uniform comprehension principle).

A stumbling block: ω-inconsistency

Restall 1992, Hajek-Paris-Shepherdson 2000, Yatabe 2005: adding ω to Ł-logic with induction schema and "ω is crisp" results into an inconsistency.
Choose R by recursion such that

$$
\begin{aligned}
k \in \Psi(x) \quad &\leftrightarrow \quad (k = 0 \otimes x \notin x) \vee \\
&\quad\ \vee (\exists n \in \omega)(k = n + 1 \otimes (x \in x \rightarrow n \in \Psi(x))) \\
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Informally $x \in R$ is equivalent to

$$x \notin x \vee (x \in x \to x \notin x) \vee (x \in x \to (x \in x \to x \notin x)) \vee \dots$$

By contraction this amount to $x \notin R$, i.e. Russell's set.

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By contraction this amount to $x \notin R$, i.e. Russell's set.

Informally $x \in R$ is equivalent to

$$x \notin x \vee (x \in x \to x \notin x) \vee (x \in x \to (x \in x \to x \notin x)) \vee \ldots$$

By contraction this amount to $x \notin R$, i.e. Russell's set.

Hence by above and by Ł-logic (IP-law!):

$$R \in R \to (\exists k \in \omega)(k \in \Psi(R))$$
$$(\exists k)(R \in R \to k \in \omega \otimes k \in \Psi(R))$$
$$(\exists k \in \omega)(R \in R \to k \in \Psi(R))$$
$$(\exists k \in \omega)(k + 1 \in \Psi(R))$$
$$(\exists k \in \omega)(k \in \Psi(R))$$
$$R \in R$$

The blue step uses $k \in \omega \vee k \notin \omega$, Indeed assume

$$R \in R \rightarrow k \in \omega \otimes k \in \Psi(R)$$

We want

$$k \in \omega \otimes (R \in R \rightarrow k \in \Psi(R))$$

If $k \in \omega$, we are done. Else, let $k \notin \omega$. Then $\neg(k \in \omega \otimes k \in \Psi(R))$ and hence $\neg R \in R$, which implies by definition $0 \in \Psi(R)$, i.e. since $0 \in \omega$,

$$(\exists k \in \omega)(k \in \Psi(R))$$

$$(\exists k \in \omega)(R \in R \rightarrow k \in \Psi(R))$$

For each $k \in \omega$, $k \notin \Psi(R)$).
By outer induction:
$k = 0$: this is simply $R \in R$, which implies $\neg\neg R \in R$, i.e. $0 \in \Psi(R)$

By IH, let $k \notin \Psi(R)$. Then $\neg(R \in R \otimes k \in \Psi(R))$, i.e. $k + 1 \notin \Psi(R)$.

Using crispness of ω and the induction rule:

$$\frac{A(0) \qquad (\forall x \in \omega)(A(x) \leftrightarrow A(x+1))}{(\forall x \in \omega)A(x)}$$

one transforms the previous argument in the derivation of a contradiction.

NB: if the induction rules is restricted to ω-free conditions, the theory is consistent (Hajek 2005).

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Induction rule vs. Induction axiom

Using the induction rule one proves:

$$(\forall x)(x \in \omega \leftrightarrow x \in \omega \otimes x \in \omega)$$

If the axiom is accepted, then one would accept for each $n \in \omega$

$$A(0) \wedge (A(0) \rightarrow A(1)) \wedge \ldots (A(n-1) \rightarrow A(n)) \rightarrow A(0) \wedge \ldots \wedge A(n)$$

which is an instance of a classical tautology which is not substructural . . .

Sharpening ω-inconsistency

Let QF-GSR$^\omega$ be the "subsystem" of GSR which (i) has pairing and projection operators as primitive with corresponding natural axioms; (ii) RAP restricted to quantifier-free formulas; (iii) ω-crispness:

$$t \in \omega \Rightarrow t \in \omega \otimes t \in \omega;$$

(iv) the IP-rule: if $x \notin FV(A)$,

$$\frac{A \Rightarrow (\exists x)B(x)}{\Rightarrow (\exists x)(A \rightarrow B(x))}$$

Theorem

QF-GSR$^\omega$ is ω-inconsistent

Proof: the recursion theorem still holds

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Theorem

QF-GSR$^{\omega}$ is ω-inconsistent

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Arithmetic in substructural logic = classical arithmetic

Fact. The class of crisp conditions is closed under elementary operations.
Hence, once $=$ is crisp, by induction one shows that every arithmetical formula is crisp!

GS-rules

- $\top$-rule:

$$\Gamma \Rightarrow \Delta, \top$$

- $\exists$-rules:

$$\frac{\Gamma, A[x := a] \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A[x := s]}{\Gamma \Rightarrow \Delta, \exists x A}$$

Proviso: $a \notin FV(\Gamma, \exists x A \Rightarrow \Delta)$.

- $\wedge$-rules ($i = 1, 2$):

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \qquad \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

- $\rightarrow$-rules:

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma', A \rightarrow B \Rightarrow \Delta, \Delta'}$$

- $\otimes$-rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma' \Rightarrow \Delta', B}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \otimes B} \qquad \frac{\Gamma, A, B, \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta}$$

- Cut:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Algebraic Preliminaries

An ML-algebra is a commutative integral bounded residuated lattice, i.e. a structure

$$\langle L, \vee, \wedge, \otimes, \rightarrow, \top, \bot \rangle$$

such that

1. $\langle L, \vee, \wedge \rangle$ is a lattice with maximum $\top$, minimum $\bot$;
2. $\langle L, \otimes, \top \rangle$ is a commutative semigroup with unit $\top$;
3. $\otimes$ and $\rightarrow$ form an adjoint pair: for all $x, y, z \in L$, $x \leq (y \rightarrow z)$ iff $x \otimes y \leq z$.

ML-algebras: semantics for the **multiplicative additive fragments of intuitionistic affine linear logic**.
Define:

$$\neg x = (x \to \top); \quad x + y = \neg(\neg x \otimes \neg y)$$

An ML-algebras is **involutive** (**linear, divisible**) if it satisfies in addition (in the given order):

1. INV: $\neg\neg x = x$;
2. LIN: $(x \to y) \vee (y \to x)$;
3. DIV: $x \wedge y = x \otimes (x \to y)$.

ML-Logics

1) IML = logic of involutive ML-algebras (Grishin);
2) MTL= logic of linear ML-algebras;
3) IMTL= logic of involutive linear ML-algebras;
4) BL= logic of divisible linear ML-algebras (Hajek);
5) Ł = logic of involutive divisible linear ML-algebras (Łukasiewicz)

NB: adding contraction $x \otimes x = x$ to BL yields the Gödel-Dummett logic, and to ML (IML) intuitionistic (classical) logic.