On contraction and the modal fragment

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Introduction

- Contraction is the reason for the undecidability of first-order logic
- If contraction is excluded, then there are no infinite paths in the proof search and thus derivability becomes decidable
- There is no syntactic characterization available for the derivable formulae of a contraction free fragment of first-order logic
- We present a one-sided sequent calculus T in which only a controlled form of contraction is available. T is complete with respect to the modal fragment of first-order logic.

Roadmap

- Introduction
- The modal fragment
- A calculus with controlled contraction
- Multi-modal logic
- Completeness

The language of the modal fragment

Definition

The formulae of $\mathcal{L}_1^M$ are defined inductively as follows.

1. If P is a unary relation symbol, then $P(u)$ and its negation $\sim P(u)$ are (atomic) $\mathcal{L}_1^M$ formulae for every variable u.
2. If A and B are $\mathcal{L}_1^M$ formulae with $\mathsf{FV}(A) = \mathsf{FV}(B)$ then $A \land B$ and $A \lor B$ are $\mathcal{L}_1^M$ formulae.
3. Let R be a binary relation symbol and $B(v)$ be an $\mathcal{L}_1^M$ formula. Then

$$\forall v(\sim R(u,v) \lor B(v)) \text{ and } \exists v(R(u,v) \land B(v))$$

are $\mathcal{L}_1^M$ formulae for every variable u which is different from v.

Note that an $\mathcal{L}_1^M$ formula contains exactly one variable free.
A sequent $\Gamma, \Delta, \ldots$ is a finite multiset of formulae.

The one-sided sequent calculus T

Axioms:

$$\Phi, P, {\sim}P \quad (Ax).$$

Propositional rules:

$$\frac{\Phi, A, B}{\Phi, A \vee B} \quad (\vee), \qquad\qquad \frac{\Phi, A \qquad\qquad \Phi, B}{\Phi, A \wedge B} \quad (\wedge),$$

Quantifier rules:

$$\frac{\Phi, B(u)}{\Phi, \forall u B(u)} \quad (\forall)$$

where we assume that the variable u does not occur free in the conclusion $\Phi, \forall u B(u)$, and

$$\frac{\Phi, B(u)}{\Phi, \exists u B(u)} \quad (\exists c).$$

Basic facts

By induction on the length of derivations, we can easily see that a weakening lemma holds for T.

Lemma

For all sequents Γ *and* Δ *we have* $\mathsf{T} \vdash \Gamma \Longrightarrow \mathsf{T} \vdash \Gamma, \Delta$.

Remark

If we replace the rule $(\exists c)$ *with the following*

$$\frac{\Phi, \exists u B(u), B(u)}{\Phi, \exists u B(u)} \quad (\exists),$$

then we obtain a system which is complete for full first-order logic.

The language of modal logic

The language $\mathcal{L}_M$ of multi-modal logic comprises countably many atomic propositions $p_1, p_2, \ldots$ and the symbols $\sim$ (atomic negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\Diamond_i$ and $\Box_i$ (modal operators) for every natural number i.

The system K for multi-modal logic

Axioms:

$$\Gamma, p, {\sim}p \quad (Ax).$$

Propositional rules:

$$\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad (\vee), \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \quad (\wedge).$$

Modal rules:

$$\frac{\Gamma, A}{\Diamond_i \Gamma, \Box_i A, \Sigma} \quad (\Box)$$

where $\Diamond_i\{B_1, \ldots, B_k\} := \{\Diamond_i B_1, \ldots, \Diamond_i B_k\}$.

Completeness of K

Theorem

The system K *is sound and complete for* $\mathcal{L}_M$ *formulae.*

Proof. We call a finite set Φ of $\mathcal{L}_M$ formulae *saturated* if

1. $\mathsf{K} \nvdash \Phi$,
2. $A \wedge B \in \Phi$ implies $A \in \Phi$ or $B \in \Phi$, and
3. $A \vee B \in \Phi$ implies $A \in \Phi$ and $B \in \Phi$.

It is easy to show that

$$\text{for each sequent } \Delta \text{ with } \mathsf{K} \nvdash \Delta \text{, there exists a saturated set } \Phi \text{ such that } \Phi \text{ is a superset of the underlying set of } \Delta. \tag{1}$$

We define the Kripke structure $\mathcal{M} = (W, R_1, \ldots, R_n, \lambda)$ as follows:

1. W consists of all saturated sets,
2. for any $\Phi, \Psi \in W$ we set $(\Phi, \Psi) \in R_i$ if $\{A : \Diamond_i A \in \Phi\} \subseteq \Psi$,
3. $\lambda(p) := \{\Phi \in W : p \notin \Phi\}$.

Completeness of K (2)

By induction on the structure of the formula A we can show that

$$\begin{aligned}&\text{for all formulae } A \text{ and all } \Phi \in W \\ &\text{we have } A \in \Phi \Rightarrow \mathcal{M}, \Phi \not\models A.\end{aligned} \tag{2}$$

We only show the case for $A = \Box_i B$. We have
K $\nvdash B, \{C : \Diamond_i C \in \Phi\}$, since otherwise by the $(\Box)$ rule we would obtain K $\vdash \Phi$ which contradicts Φ saturated. By (1) there exists Ψ saturated with $B, \{C : \Diamond_i C \in \Phi\} \subseteq \Psi$. By the induction hypothesis we obtain $\mathcal{M}, \Psi \not\models B$. The definition of R_i gives us $(\Phi, \Psi) \in R_i$. Hence we conclude $\mathcal{M}, \Phi \not\models \Box_i B$.
To obtain completeness of K assume K $\nvdash A$ for some formula A. By (1) there exits a saturated set Φ which contains A. By (2) we find $\mathcal{M}, \Phi \not\models A$. Thus A is not valid.

The standard translation $\mathsf{ST}_u(\cdot)$

1. $\mathsf{ST}_u([\sim]p_i) := [\sim]P_i(u)$,
2. $\mathsf{ST}_u(A * B) := \mathsf{ST}_u(A) * \mathsf{ST}_u(B)$ for $* \in \{\vee, \wedge\}$,
3. $\mathsf{ST}_u(\Box_i A) := \forall v(\sim R_i(u, v) \vee \mathsf{ST}_v(A))$,
4. $\mathsf{ST}_u(\Diamond_i A) := \exists v(R_i(u, v) \wedge \mathsf{ST}_v(A))$,
5. $\mathsf{ST}_v([\sim]p_i) := [\sim]P_i(v)$,
6. $\mathsf{ST}_v(A * B) := \mathsf{ST}_v(A) * \mathsf{ST}_v(B)$ for $* \in \{\vee, \wedge\}$,
7. $\mathsf{ST}_v(\Box_i A) := \forall u(\sim R_i(v, u) \vee \mathsf{ST}_u(A))$,
8. $\mathsf{ST}_v(\Diamond_i A) := \exists u(R_i(v, u) \wedge \mathsf{ST}_u(A))$.

where v is a variable different from u.
For a sequent $\Phi = A_1, \ldots, A_n$ of $\mathcal{L}_M$ formulae, we define $\mathsf{ST}_u(\Phi) = \mathsf{ST}_u(A_1), \ldots, \mathsf{ST}_u(A_n)$.

Embedding

Remark

If we identify $\mathcal{L}_1^M$ *formulae that differ only in the names of bound variables (whether an* $\mathcal{L}_1^M$ *formula is provable in* T *does not depend on the names of its bound variables), then each* $\mathcal{L}_1^M$ *formula* $A(u)$ *is the standard translation* $\mathsf{ST}_u(C)$ *of some* $\mathcal{L}_M$ *formula* C*, and conversely, for each* $\mathcal{L}_M$ *formula* C*,* $\mathsf{ST}_u(C)$ *is an* $\mathcal{L}_1^M$ *formula.*

Lemma

Let Φ *be a sequent of* $\mathcal{L}_M$ *formulae. Then*

$$\mathsf{K} \overset{n}{\vdash} \Phi \quad \Longrightarrow \quad \mathsf{T} \vdash \mathsf{ST}_u(\Phi).$$

for each variable u *of* $\mathcal{L}_1$*.*

Proof by induction on n

Case ($\Box$): let $\Phi := \Diamond_i \Psi, \Box_i A, \Xi$ and $\Gamma := \mathsf{ST}_u(\Phi)$ which is then of the form

$$\exists v(R_i(u,v) \wedge B_1(v)), \dots, \exists v(R_i(u,v) \wedge B_k(v)), \forall v(\sim R_i(u,v) \vee C(v)), \Sigma.$$

By I.H. we get $\mathsf{T} \vdash B_1(v), \dots, B_k(v), C(v)$. Weakening yields $\mathsf{T} \vdash B_1(v), \dots, B_k(v), \sim R_i(u,v), C(v)$. Now consider the following derivation in T where $\Delta := B_2(v), \dots, B_k(v)$.

$$\dfrac{\dfrac{\dots \qquad \dfrac{\dfrac{R_i(u,v), \Delta, \sim R_i(u,v), C(v) \qquad B_1(v), \Delta, \sim R_i(u,v), C(v)}{R_i(u,v) \wedge B_1(v), \Delta, \sim R_i(u,v), C(v)}}{\exists v(R_i(u,v) \wedge B_1(v)), \Delta, \sim R_i(u,v), C(v)}}{\vdots}}{\dfrac{\dfrac{\exists v(R_i(u,v) \wedge B_1(v)), \dots, \exists v(R_i(u,v) \wedge B_k(v)), \sim R_i(u,v), C(v)}{\exists v(R_i(u,v) \wedge B_1(v)), \dots, \exists v(R_i(u,v) \wedge B_k(v)), \sim R_i(u,v) \vee C(v)}}{\exists v(R_i(u,v) \wedge B_1(v)), \dots, \exists v(R_i(u,v) \wedge B_k(v)), \forall v(\sim R_i(u,v) \vee C(v))}}$$

Again applying weakening yields $\mathsf{T} \vdash \Gamma$ which finishes our proof.

Completeness

Theorem

For each $\mathcal{L}_1^M$ *formula* $A(u)$,

$$\mathsf{T} \vdash A(u) \iff \models A(u).$$

Proof.

The direction from left to right is the standard soundness result. To show the direction from right to left assume that $A(u)$ is valid. There is an $\mathcal{L}_M$ formula B such that $\mathsf{ST}_u(B) = A(u)$ (modulo renaming of bound variables). Thus $\mathsf{ST}_u(B)$ is a valid $\mathcal{L}_1^M$ formula and therefore, B is valid with respect to the Kripke semantics. Completeness of K gives us $\mathsf{K} \vdash B$. By the above lemma, we finally conclude $\mathsf{T} \vdash A(u)$. □

Conclusion

- Proof-theoretic answer to the question about the robust decidability of modal logics
- Complements the model-theoretic and automata-theoretic point of view on this issue
- Controlled contraction provides a better explanation than the two variable fragment
- We obtain that K_n is in $\mathrm{PS}\textsc{pace}$
- Is there a syntactic characterization of formulae provable without contraction?
- Can we characterize guarded fragments in terms of some restriction of contraction?

Thank you!